

Exact algorithm for the Maximum Induced Planar Subgraph Problem

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Outline of the result

Theorem

There is an $O(1.73601^n)$ algorithm for the MAX INDUCED PLANAR SUBGRAPH problem.

Ingredients:

1. [Fomin, Villanger, STACS 2010]: an $O(1.73601^n \cdot n^{t+3})$ algorithm for the MAX INDUCED SUBGRAPH OF TREEWIDTH $\leq t$
2. [Robertson, Seymour 1986; Fomin, Thilikos 2006]: planar graphs have treewidth $O(\sqrt{n})$

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3. Combinatorial results on **minimal triangulations** and **potential maximal cliques** of planar graphs.
4. An algorithm putting everything together

Motivation and related work

Exact algorithms for NP-hard problems

- [Gödel, 1959]: "how strongly in general the number of steps in finite combinatorial problems can be reduced with respect to simple exhaustive search?"
- Nice combinatorics, nice algorithmic techniques

MAX INDUCED SUBGRAPH WITH PROPERTY Π

- MAX INDEPENDENT SET [Moon, Moser, 1965; Fomin, Grandoni, Kratsch 2009]
- MAX FEEDBACK VERTEX SET [Razgon 2006; Fomin, Villanger 2010]
- MAX INDUCED SUBGRAPH OF TREEWIDTH $\leq t$ [Fomin, Villanger 2010]

MAX INDUCED SUBGRAPH OF $tw \leq t$

[Fomin, Villanger 2010]

F an induced subgraph of G , H_F a **minimal triangulation** of F .

- There is a minimal triangulation H_G of G such that H_F is an induced subgraph of G ; we say that H_F and H_G are **compatible**

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- **Fix** a minimal triangulation H_G . One can compute a **maximum induced subgraph** F of G s.t. $tw(F) \leq t$ and F has an optimal triangulation **compatible** with H_G in $O(n^{t+cst})$ time

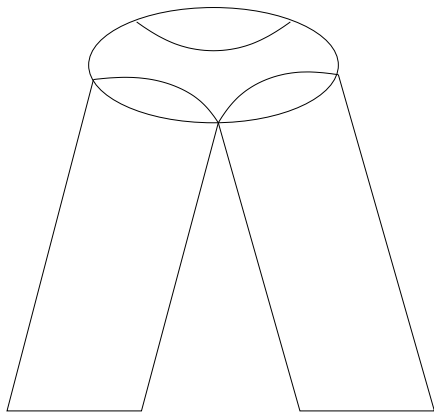
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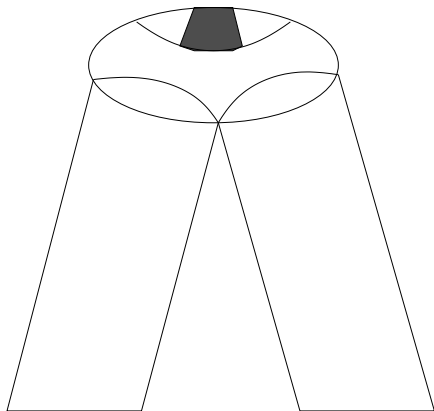
- There is a minimal triangulation H_G of G such that H_F is an induced subgraph of G ; we say that H_F and H_G are **compatible**
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- **Fix** a minimal triangulation H_G . One can compute a **maximum induced subgraph** F of G s.t. $tw(F) \leq t$ and F has an optimal triangulation **compatible** with H_G in $O(n^{t+cst})$ time
- Dynamic programming over all minimal triangulations H_G using **potential maximal cliques**: $O(1.73601^n \cdot n^{t+cst})$ time

Computing F for a fixed tree decomposition of G



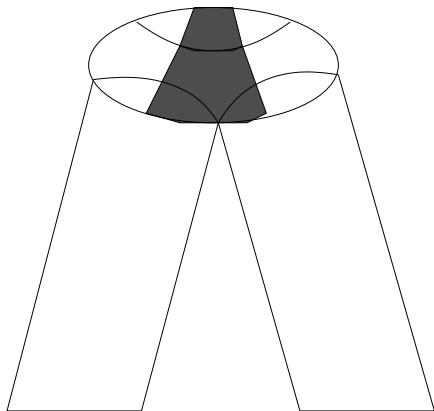
- $\alpha(S, W, C)$: the size of the largest partial solution intersecting S in W

Computing F for a fixed tree decomposition of G



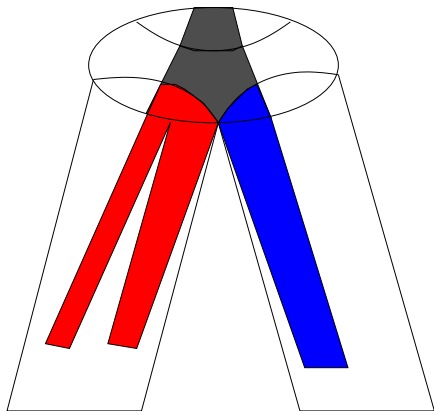
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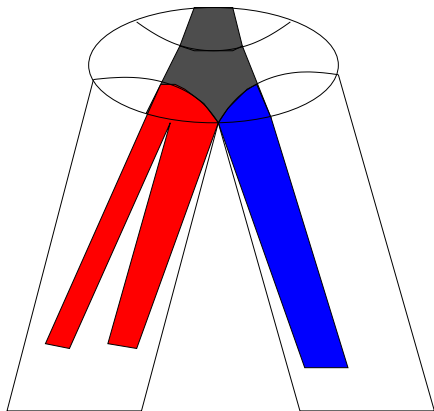
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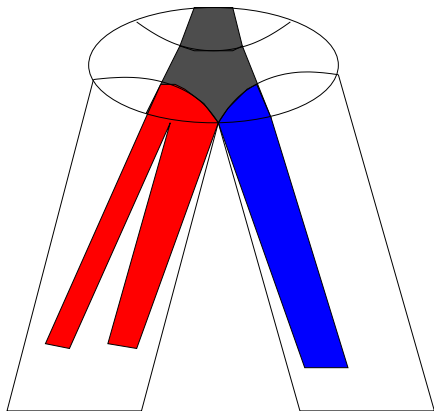
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- Running time: $O(n^{t+cst})$
- Also constructs a minimal triangulation of F

Browsing through all minimal triangulations of G

Definition

A vertex subset Ω_G of G is a **potential maximal clique** if there exists a minimal triangulation H_G such that Ω_G is a maximal clique of H_G .

- One can "browse" through all minimal triangulations of a graph, in time $O^*(\#p.m.c.)$ [Bouchitté, Todinca 2001, Fomin, Kratsch, Todinca 2008]
- An n -vertex graph has $O^*(1.73601^n)$ potential maximal cliques [Fomin Villanger 2010]

A MAXIMUM INDUCED SUBGRAPH OF $tw \leq t$ can be computed in $O(1.73601^n \cdot n^{t+cst})$ time [Fomin Villanger 2010].

Towards an extension to the MAX INDUCED PLANAR GRAPH problem

- Good news: planar graphs have treewidth at most $3.182\sqrt{n}$ [Fomin, Thilikos 2006];
an $O(\#p.m.c. \cdot n^{c\sqrt{n}+cst}) = O(\#p.m.c. \cdot 2^{o(n)}) = O(1.73601^n)$ algorithm?

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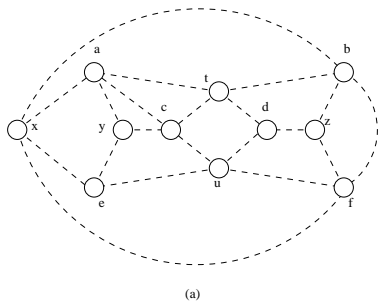
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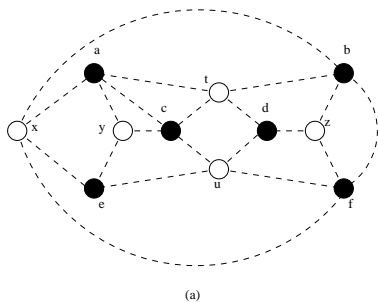
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We need more tools for gluing partial solutions. Recall that we glue along **potential maximal cliques** of the target (planar) graph F .

Potential maximal cliques in planar graphs

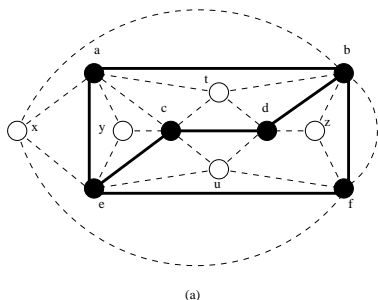


Potential maximal cliques in planar graphs



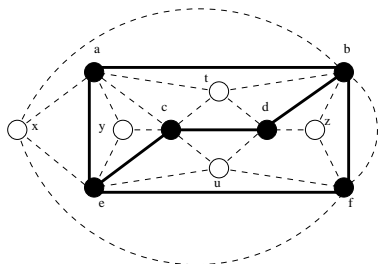
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Potential maximal cliques in planar graphs

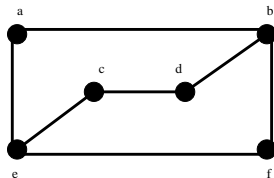


- A **potential maximal clique** Ω_F of a plane graph F forms a **θ -structure** [Bouchitté, Mazoit, Todinca 2003; ...]

Potential maximal cliques in planar graphs



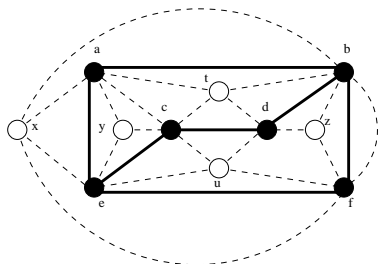
(a)



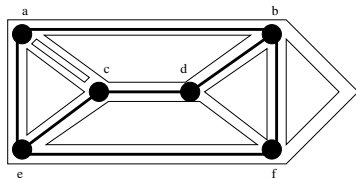
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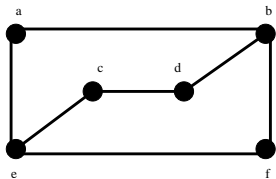
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- A **potential maximal clique** Ω_F of a plane graph F forms a **θ -structure** [Bouchitté, Mazoit, Todinca 2003; ...]
- The neighbourhoods of the components of $F - \Omega_F$ correspond to **pairwise non-crossing** Jordan curves
- Conversely, if we draw planar pieces "inside" these curves we preserve planarity

θ -structures and neighborhood assignments



$$\mathcal{S}_1 = [e, a, b, f]$$

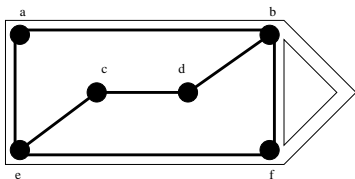
$$\mathcal{S}_2 = [e, c, d, b, f]$$

$$\mathcal{S}_3 = [e, a, b, d, c]$$

We can avoid geometry.

- θ -structures $\theta(\Omega_F)$ on Ω_F : three totally ordered subsets, sharing the extremities and forming three cyclic orderings \mathcal{S}_i .
- **neighborhood assignment** $[\theta(\Omega_F)]$:
 - assigns to each cyclic ordering \mathcal{S}_i of $\theta(\Omega_F)$ a set of pairwise non-crossing subsets of \mathcal{S}_i .
 - for each edge xy of $F[\Omega_F]$, the neighborhood $\{x, y\}$ is assigned to some \mathcal{S}_i .

θ -structures and neighborhood assignments

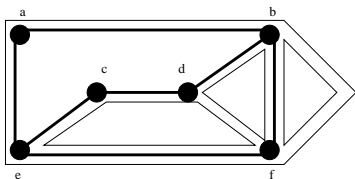


$$\mathcal{S}_1 : \{b, f\}, \{a, b, f\}$$

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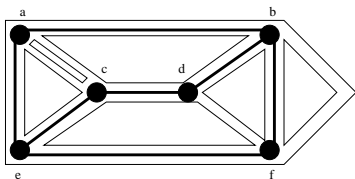


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θ -structures and neighborhood assignments



$$\mathcal{S}_3 : \\ \{a, c, e\}, \{a, c\}, \{a, b, d, c\}$$

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Counting neighborhood assignments

Theorem

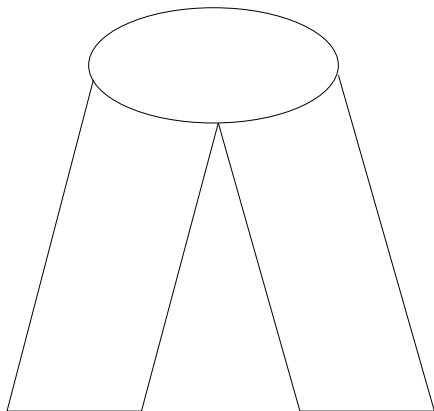
Over all subsets Ω_F of size at most $c\sqrt{n}$, over all possible θ -structures $\theta(\Omega_F)$, there are $2^{o(n)}$ possible (partial) neighborhood assignments.

- there are $\binom{n}{c\sqrt{n}}$ possible subsets Ω_F
- for each Ω_F , there are $2^{o(n)}$ possible θ -structures
- for a fixed θ -structure $\theta(\Omega_F)$, for each cyclic ordering \mathcal{S}_i the number of possible neighborhood assignments on \mathcal{S}_i is upper bounded by the Catalan number $CN(|\mathcal{S}_i|) \leq 4^{|\mathcal{S}_i|}$ [Kreweras 1972].

The algorithm

Maximum partial solution for

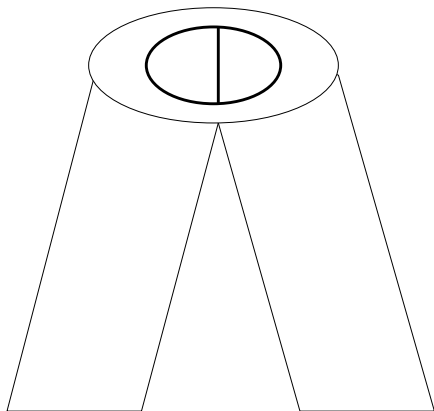
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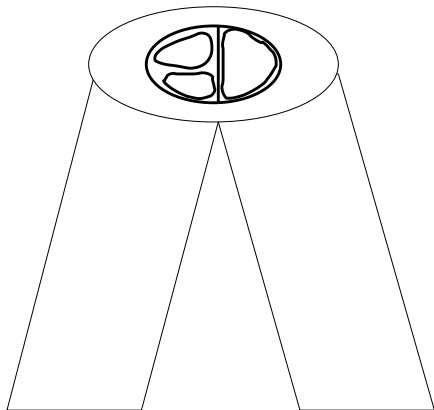
- (S, C, Ω_G)
- a θ -structure $\theta(\Omega_F)$



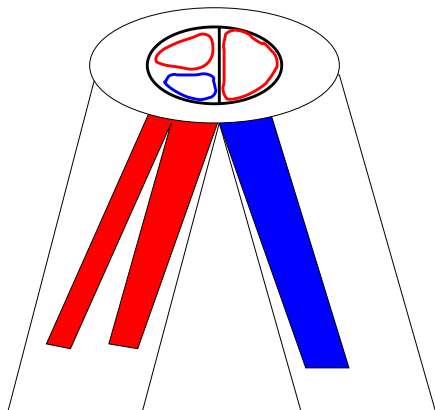
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The algorithm

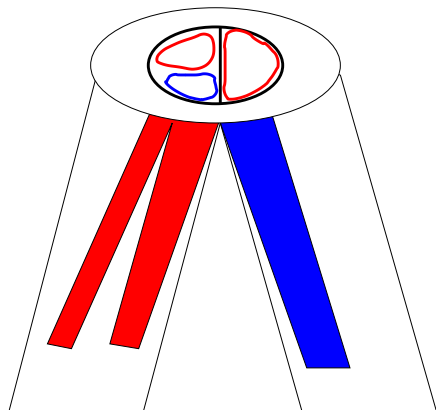


Maximum partial solution for

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Dynamic programming over all p.m.c. Ω_G of G and on all possible (partial) neighborhood assignments $[\theta(\Omega_F)]$, over all "small" subsets Ω_F .

The algorithm



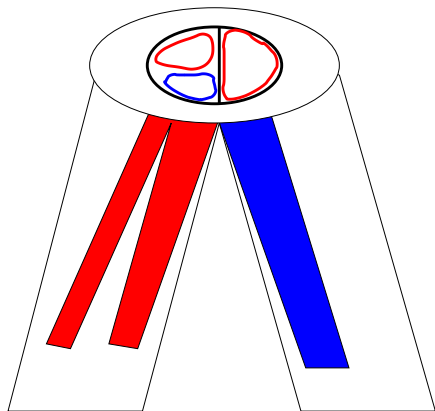
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Running time: $O(\#p.m.c. \cdot (\#\text{neighborhood assignments})^3) = O(1.73601^n)$

Conclusion and open questions

An $O(1.73601^n)$ algorithm for the MAX INDUCED PLANAR SUBGRAPH problem.

- MAX INDUCED SUBGRAPH WITH PROPERTY Π ?
 - Bounded genus?
 - Excluded minors?
 - Bounded degeneracy?
- Combinatorial questions
 - What is the maximum number of **minimal separators** in an n -vertex graph?
 - The same for potential maximal cliques? (The current upper bound $O(1.73601^n)$ does not seem tight).

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- Thank you! **Your questions?**