Split vs Modular Decomposition
(the case of totally decomposable graphs)

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Joint work with E. Gioan (CNRS LIRMM)
1 Definitions and preliminaries

2 Vertex modification of totally decomposable graphs

3 Conclusion and on-going work
A graph-labelled tree is a pair \((T, \mathcal{F})\) with \(T\) a tree and \(\mathcal{F}\) a set of graphs such that:

- each node \(v\) of degree \(k\) of \(T\) is labelled by a graph \(G_v \in \mathcal{F}\) on \(k\) vertices
- there is a bijection \(\rho_v\) from the tree-edges incident to \(v\) to the vertices of \(G_v\)
A rooted graph-labelled tree is a pair \((T, \mathcal{F})\) with \(T\) a rooted tree and \(\mathcal{F}\) a set of graphs such that:

- Each node \(v\) with \(k\) children of \(T\) is labelled by a graph \(G_v \in \mathcal{F}\) on \(k\) vertices.
- There is a bijection \(\rho_v\) from the tree-edges between \(v\) and its children to the vertices of \(G_v\).
Given a rooted graph labelled tree \((T, \mathcal{F})\), the graph \(G_M(T, \mathcal{F})\) has the leaves of \(T\) as vertices and

- \(xy \in E(G_S(T, \mathcal{F}))\) iff \(\rho_v(\rho(v)u)\rho(v)w \in E(G_v)\), \(uv, vw\) tree-edges on the \(x, y\)-path in \(T\) and \(v = lca_T(x, y)\).
Given a rooted graph labelled tree \((T, \mathcal{F})\), the graph \(G_M(T, \mathcal{F})\) has the leaves of \(T\) as vertices and

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where \(uv, vw\) are tree-edges on the \(x, y\)-path in \(T\) and \(v = lca_T(x, y)\).
A subset of vertices $M$ of a graph $G = (V, E)$ is a **module** iff
\[ \forall x \in V \setminus M, \text{ either } M \subseteq N(x) \text{ or } M \cap N(x) = \emptyset \]
Given a graph labelled tree \((T, \mathcal{F})\), the graph \(G_S(T, \mathcal{F})\) has the leaves of \(T\) as vertices and

\[ xy \in E(G_S(T, \mathcal{F})) \text{ iff } \rho_v(uv) \rho_v(vw) \in E(G_v), \forall \text{ tree-edges } uv, vw \text{ on the } x, y\text{-path in } T \]
Given a graph labelled tree \((T, \mathcal{F})\), the graph \(G_S(T, \mathcal{F})\) has the leaves of \(T\) as vertices and

\[ xy \in E(G_S(T, \mathcal{F})) \iff \rho_v(\,uv\,)\rho_v(\,vw\,) \in E(G_v), \]

\( \forall \) tree-edges \(uv, vw\) on the \(x, y\)-path in \(T\)
A bipartition \((A, B)\) of the vertices of a graph \(G = (V, E)\) is a **split** iff

- \(|A| \geq 2, |B| \geq 2\);
- for \(x \in A\) and \(y \in B\), \(xy \in E\) iff \(x \in N(B)\) and \(y \in N(A)\).
**Modular decomposition**

**Examples of modules**
- any subset of vertices of the clique
- any subset of vertices of the stable

**Split decomposition**

**Examples of splits**
- any non-trivial bipartition of the clique
- any non-trivial bipartition of the $K_{1,n}$
Definitions and preliminaries
- Vertex modification of totally decomposable graphs
- Conclusion and on-going work

**Modular decomposition**

**Examples of modules**
- any subset of vertices of the clique
- any subset of vertices of the stable

**Totally decomposable graphs**
- Cographs

**Split decomposition**

**Examples of splits**
- any non-trivial bipartition of the clique
- any non-trivial bipartition of the $K_{1,n}$

**Totally decomposable graphs**
- Distance hereditary graphs
Gallai’67 reformulated

For any graph $G$, there exists a unique rooted graph-labelled tree $(T, F)$ with a minimum number of nodes such that

1. $G = G_M(T, F)$ and
2. any graph of $F$ is prime or degenerate for the modular decomposition.
Gallai’67 reformulated

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→ We note $(T, F) = ST(G)$
1 Definitions and preliminaries

2 Vertex modification of totally decomposable graphs

3 Conclusion and on-going work
Theorem (Corneil, Pearl and Stewart '85, Sharan and Shamir '04)

Let $G = (V, E)$ be a cograph. It can be tested in

- $O(|S|)$ whether $G + (x, S)$, with $x \notin E$ and $N(x) = S$, is a cograph;
- $O(|S|)$ whether $G - x$, with $S = N(x)$, is a cograph;
- $O(1)$ whether $G + e$, with $e \notin E$, is a cograph;
- $O(1)$ whether $G - e$, with $e \in E$, is a cograph.
Theorem (Tedder and Corneil ’06, Gioan and Paul ’07)

Let $G = (V, E)$ be a distance hereditary (DH) graph. It can be tested in

- $O(|S|)$ whether $G + (x, S)$, with $x \notin E$ and $N(x) = S$, is a DH graph;
- $O(|S|)$ whether $G - x$, with $S = N(x)$, is a DH graph;
- $O(1)$ whether $G + e$, with $e \notin E$, is a DH graph;
- $O(1)$ whether $G - e$, with $e \in E$, is a DH graph.
Let \( (T, \mathcal{F}) \) be a graph-labelled tree, and \( S \) be a subset of leaves of \( T \). A node \( u \) of \( T(S) \) is:

- **fully-marked** by \( S \) if any subtree of \( T - u \) contains a leaf of \( S \);

- **singly-marked** by \( S \) if it is a star-node and exactly two subtrees of \( T - u \) contain a leaf \( l \in S \) among which the subtree containing the neighbor \( v \) of \( u \) such that \( \rho_{u}(uv) \) is the centre of \( G_u \) is the centre of \( G_u \);
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Let \((T, F)\) be a graph-labelled tree, and \(S\) be a subset of leaves of \(T\). A node \(u\) of \(T(S)\) is:

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- **partially-marked** otherwise

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**Diagram:**

A graph with nodes labeled and shaded to illustrate the marking definitions.
Theorem (DH incremental characterization [Gioan, Paul ’07] )

Let $G$ be a connected DH graph and $ST(G) = (T, F)$ be its split tree. Then $G + (x, S)$ is a DH graph iff:

1. At most one node of $T(S)$ is partially-marked.
Theorem (DH incremental characterization [Gioan, Paul '07])

Let $G$ be a connected DH graph and $ST(G) = (T, \mathcal{F})$ be its split tree. Then $G + (x, S)$ is a DH graph iff:

1. At most one node of $T(S)$ is partially-marked.
2. Any clique node of $T(S)$ is either fully or partially-marked.
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1. At most one node of $T(S)$ is partially-marked.
2. Any clique node of $T(S)$ is either fully or partially-marked.
3. If there exists a partially-marked node $u$, then any star node $v \neq u$ of $T(S)$ is oriented towards $u$ if and only if it is fully-marked.
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3. If there exists a partially-marked node $u$, then any star node $v \neq u$ of $T(S)$ is oriented towards $u$ if and only if it is fully-marked.
4. Otherwise, there exists a tree-edge $e$ of $T(S)$ towards which any star node of $T(S)$ is oriented if and only if it is fully-marked.
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Split vs Modular Decomposition (the case of totally decomposable graphs)
The insertion fails: the two singly-marked nodes are oriented towards the partially-marked node!
The insertion succeeds: in $G_S(T, \mathcal{F})$, we have $N(x) = S$.
Insertion algorithm

1. Extract $T(S)$ (require an arbitrary orientation of $ST(G)$);
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2. Check the mark-type of the nodes and look for an insertion node or edge;
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1. Extract $T(S)$ (require an arbitrary orientation of $ST(G)$);
2. Check the mark-type of the nodes and look for an insertion node or edge;
3. Insert the node by either subdividing the insertion edge, or splitting the insertion node, or attaching $x$ to the insertion node.
Cographs

Let \((T, \mathcal{F})\) be a rooted graph-labelled tree, and \(S\) be a subset of leaves of \(T\). A node \(u\) of \(T(S)\) is:

- **fully-marked** by \(S\) if for any children \(v\) of \(u\), \(T_v\) contains some leaves of \(S\);

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- **partially-marked** otherwise.
Cographs

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Theorem (Cograph incremental characterization [CPS’85] )

Let $G$ be a cograph and $MD(G) = (T, \mathcal{F})$ be its modular decomposition tree. Then $G + (x, S)$ is a cograph iff:

1. At most one node of $T(S)$ is partially-accessible.
Theorem (Cograph incremental characterization [CPS’85] )

Let $G$ be a cograph and $MD(G) = (T, F)$ be its modular decomposition tree. Then $G + (x, S)$ is a cograph iff:

1. At most one node of $T(S)$ is partially-accessible.
2. Any series node of $T(S)$ is either fully or partially-accessible.
Theorem (Cograph incremental characterization [CPS’85] )

Let $G$ be a cograph and $MD(G) = (T, F)$ be its modular decomposition tree. Then $G + (x, S)$ is a cograph iff:

1. At most one node of $T(S)$ is partially-accessible.
2. Any series node of $T(S)$ is either fully or partially-accessible.
3. If there exists a partially-accessible node $u$, then a parallel node $v \neq u$ of $T(S)$ is a descendant of $u$ if and only if it is fully-accessible.
Theorem (Cograph incremental characterization [CPS'85])

Let \( G \) be a cograph and \( MD(G) = (T, \mathcal{F}) \) be its modular decomposition tree. Then \( G + (x, S) \) is a cograph iff:

1. At most one node of \( T(S) \) is partially-accessible.
2. Any series node of \( T(S) \) is either fully or partially-accessible.
3. If there exists a partially-accessible node \( u \), then a parallel node \( v \neq u \) of \( T(S) \) is a descendant of \( u \) if and only if it is fully-accessible.
4. Otherwise, there exists a tree-edge \( e = uw \) of \( T(S) \) such that a parallel node \( v \neq u \) of \( T(S) \) is a descendant of \( u \) if and only if it is fully-accessible.
1. Definitions and preliminaries

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Remarks

- There exists a characterization of cographs in terms of split tree.
- The graph-labelled tree representation of DH graphs yields an intersection model characterizing DH graphs.
- The edge-modification seems also to be possible using the split tree representation.
On-going work

- **Circle graphs and permutation graphs**: properties similar than for prime permutation graphs are observed for prime circle graphs.

- **Computation of the split tree**: An algorithm scheme like Ehrenfeucht et al’s one for the modular decomposition can be design for the split decomposition.

- **Factorizing permutation**: such a concept would play the same role as for modular decomposition.

- **Generalization to other decompositions**: apply this graph-labelled tree approach to other decompositions.