

SPLIT VS MODULAR DECOMPOSITION

(the case of totally decomposable graphs)

Christophe Paul

CNRS - LIRMM - Université Montpellier II, France

March 29, 2007

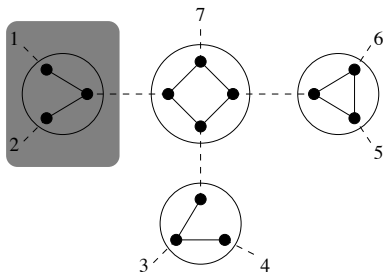
Joint work with E. Gioan (CNRS LIRMM)

- 1 Definitions and preliminaries
- 2 Vertex modification of totally decomposable graphs
- 3 Conclusion and on-going work

Graph labeled tree

A *graph-labelled tree* is a pair (T, \mathcal{F}) with T a tree and \mathcal{F} a set of graphs such that:

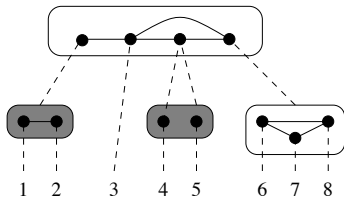
- each node v of degree k of T is labelled by a graph $G_v \in \mathcal{F}$ on k vertices
- there is a bijection ρ_v from the tree-edges incident to v to the vertices of G_v



Rooted graph labeled tree

A *rooted graph-labelled tree* is a pair (T, \mathcal{F}) with T a rooted tree and \mathcal{F} a set of graphs such that:

- each node v with k children of T is labelled by a graph $G_v \in \mathcal{F}$ on k vertices
- there is a bijection ρ_v from the tree-edges between v and its children to the vertices of G_v

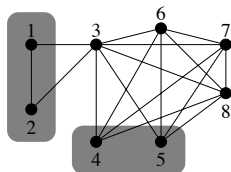
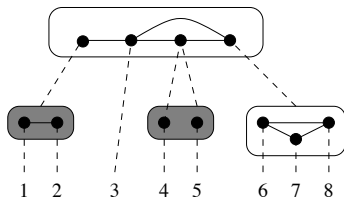


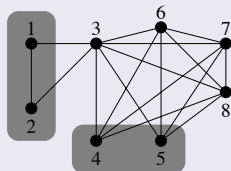
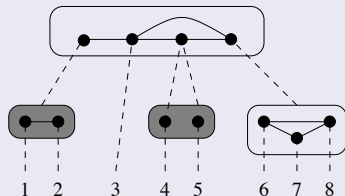
Given a rooted graph labelled tree (T, \mathcal{F}) , the graph $G_M(T, \mathcal{F})$ has the leaves of T as vertices and

- $xy \in E(G_S(T, \mathcal{F}))$ iff $\rho_v(uv)\rho_v(vw) \in E(G_v)$,
 uv, vw tree-edges on the x, y -path in T and $v = lca_T(x, y)$.

Given a rooted graph labelled tree (T, \mathcal{F}) , the graph $G_M(T, \mathcal{F})$ has the leaves of T as vertices and

- $xy \in E(G_S(T, \mathcal{F}))$ iff $\rho_v(uv)\rho_v(vw) \in E(G_v)$,
 uv, vw tree-edges on the x, y -path in T and $v = lca_T(x, y)$.





Modules

A subset of vertices M of a graph $G = (V, E)$ is a **module** iff

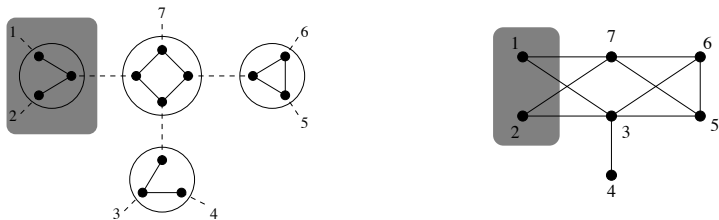
$$\forall x \in V \setminus M, \text{ either } M \subseteq N(x) \text{ or } M \cap N(x) = \emptyset$$

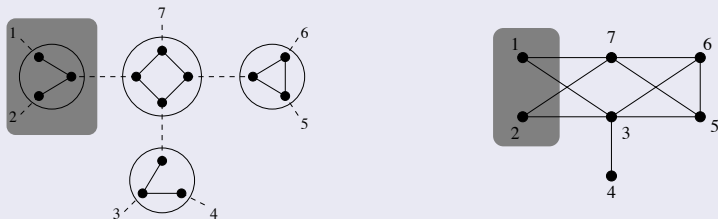
Given a graph labelled tree (T, \mathcal{F}) , the graph $G_S(T, \mathcal{F})$ has the leaves of T as vertices and

- $xy \in E(G_S(T, \mathcal{F}))$ iff $\rho_v(uv)\rho_v(vw) \in E(G_v)$,
 \forall tree-edges uv, vw on the x, y -path in T

Given a graph labelled tree (T, \mathcal{F}) , the graph $G_S(T, \mathcal{F})$ has the leaves of T as vertices and

- $xy \in E(G_S(T, \mathcal{F}))$ iff $\rho_v(uv)\rho_v(vw) \in E(G_v)$,
 \forall tree-edges uv, vw on the x, y -path in T





Split

A bipartition (A, B) of the vertices of a graph $G = (V, E)$ is a **split** iff

- $|A| \geq 2$, $|B| \geq 2$;
- for $x \in A$ and $y \in B$, $xy \in E$ iff $x \in N(B)$ and $y \in N(A)$.

Modular decomposition

Examples of modules

- any subset of vertices of the clique
- any subset of vertices of the stable

Split decomposition

Examples of splits

- any non-trivial bipartition of the clique
- any non-trivial bipartition of the $K_{1,n}$

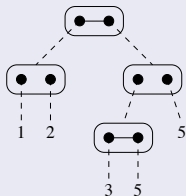
Modular decomposition

Examples of modules

- any subset of vertices of the clique
- any subset of vertices of the stable

Totally decomposable graphs

- Cographs



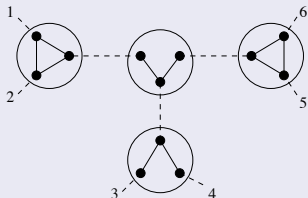
Split decomposition

Examples of splits

- any non-trivial bipartition of the clique
- any non-trivial bipartition of the $K_{1,n}$

Totally decomposable graphs

- Distance hereditary graphs



Gallai'67 reformulated

For any graph G , there exists a unique rooted graph-labelled tree (T, \mathcal{F}) with a minimum number of nodes such that

- 1 $G = G_M(T, \mathcal{F})$ and
- 2 any graph of \mathcal{F} is prime or degenerate for the modular decomposition.

Gallai'67 reformulated

For any graph G , there exists a unique rooted graph-labelled tree (T, \mathcal{F}) with a minimum number of nodes such that

- 1 $G = G_M(T, \mathcal{F})$ and
- 2 any graph of \mathcal{F} is prime or degenerate for the modular decomposition.

Cunningham'82 reformulated

For any connected graph G , there exists a unique graph-labelled tree (T, \mathcal{F}) with a minimum number of nodes such that

- 1 $G = G_S(T, \mathcal{F})$,
- 2 any graph of \mathcal{F} is prime or degenerate for the split decomposition.

Gallai'67 reformulated

For any graph G , there exists a unique rooted graph-labelled tree (T, \mathcal{F}) with a minimum number of nodes such that

- ① $G = G_M(T, \mathcal{F})$ and
- ② any graph of \mathcal{F} is prime or degenerate for the modular decomposition.

→ We note $(T, \mathcal{F}) = MD(G)$

Cunningham'82 reformulated

For any connected graph G , there exists a unique graph-labelled tree (T, \mathcal{F}) with a minimum number of nodes such that

- ① $G = G_S(T, \mathcal{F})$,
- ② any graph of \mathcal{F} is prime or degenerate for the split decomposition.

→ We note $(T, \mathcal{F}) = ST(G)$

- 1 Definitions and preliminaries
- 2 Vertex modification of totally decomposable graphs
- 3 Conclusion and on-going work

Theorem (Corneil, Pearl and Stewart '85, Sharan and Shamir '04)

Let $G = (V, E)$ be a cograph. It can be tested in

- $O(|S|)$ whether $G + (x, S)$, with $x \notin E$ and $N(x) = S$, is a cograph;
- $O(|S|)$ whether $G - x$, with $S = N(x)$, is a cograph;
- $O(1)$ whether $G + e$, with $e \notin E$, is a cograph;
- $O(1)$ whether $G - e$, with $e \in E$, is a cograph.

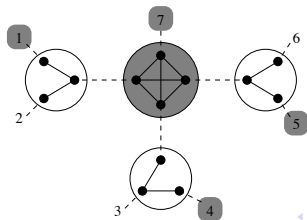
Theorem (Tedder and Corneil '06, Gioan and Paul '07)

Let $G = (V, E)$ be a distance hereditary (DH) graph. It can be tested in

- $O(|S|)$ whether $G + (x, S)$, with $x \notin E$ and $N(x) = S$, is a DH graph;
- $O(|S|)$ whether $G - x$, with $S = N(x)$, is a DH graph;
- $O(1)$ whether $G + e$, with $e \notin E$, is a DH graph;
- $O(1)$ whether $G - e$, with $e \in E$, is a DH graph.

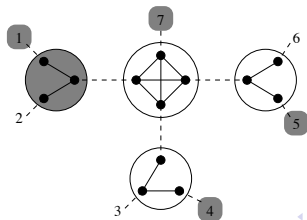
Let (T, \mathcal{F}) be a graph-labelled tree, and S be a subset of leaves of T . A node u of $T(S)$ is:

- **fully-marked** by S if any subtree of $T - u$ contains a leaf of S ;



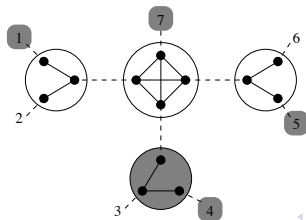
Let (T, \mathcal{F}) be a graph-labelled tree, and S be a subset of leaves of T . A node u of $T(S)$ is:

- **fully-marked** by S if any subtree of $T - u$ contains a leaf of S ;
- **singly-marked** by S if it is a star-node and exactly two subtrees of $T - u$ contain a leaf $l \in S$ among which the subtree containing the neighbor v of u such that $\rho_u(uv)$ is the centre of G_u ;



Let (T, \mathcal{F}) be a graph-labelled tree, and S be a subset of leaves of T . A node u of $T(S)$ is:

- **fully-marked** by S if any subtree of $T - u$ contains a leaf of S ;
- **singly-marked** by S if it is a star-node and exactly two subtrees of $T - u$ contain a leaf $l \in S$ among which the subtree containing the neighbor v of u such that $\rho_u(uv)$ is the centre of G_u ;
- **partially-marked** otherwise



Theorem (DH incremental characterization [Gioan, Paul '07])

Let G be a connected DH graph and $ST(G) = (T, \mathcal{F})$ be its split tree. Then $G + (x, S)$ is a DH graph iff:

- 1 At most one node of $T(S)$ is partially-marked.

Theorem (DH incremental characterization [Gioan, Paul '07])

Let G be a connected DH graph and $ST(G) = (T, \mathcal{F})$ be its split tree. Then $G + (x, S)$ is a DH graph iff:

- 1 At most one node of $T(S)$ is partially-marked.
- 2 Any clique node of $T(S)$ is either fully or partially-marked.

Theorem (DH incremental characterization [Gioan, Paul '07])

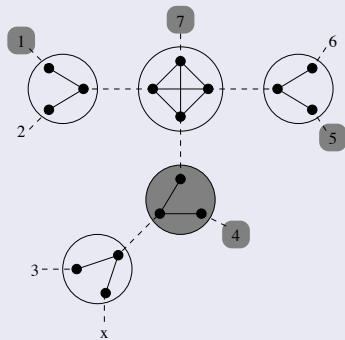
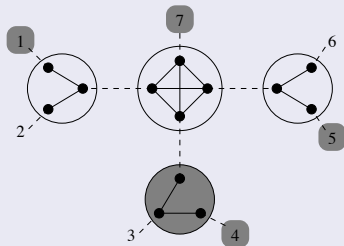
Let G be a connected DH graph and $ST(G) = (T, \mathcal{F})$ be its split tree. Then $G + (x, S)$ is a DH graph iff:

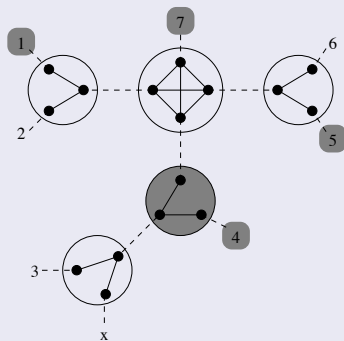
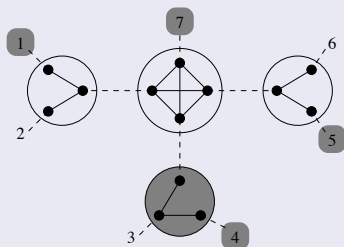
- 1 At most one node of $T(S)$ is partially-marked.
- 2 Any clique node of $T(S)$ is either fully or partially-marked.
- 3 If there exists a partially-marked node u , then any star node $v \neq u$ of $T(S)$ is oriented towards u if and only if it is fully-marked.

Theorem (DH incremental characterization [Gioan, Paul '07])

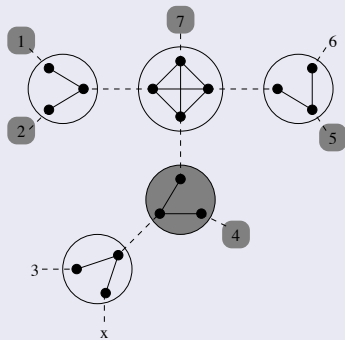
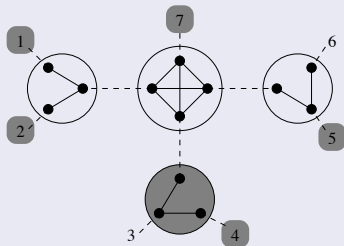
Let G be a connected DH graph and $ST(G) = (T, \mathcal{F})$ be its split tree. Then $G + (x, S)$ is a DH graph iff:

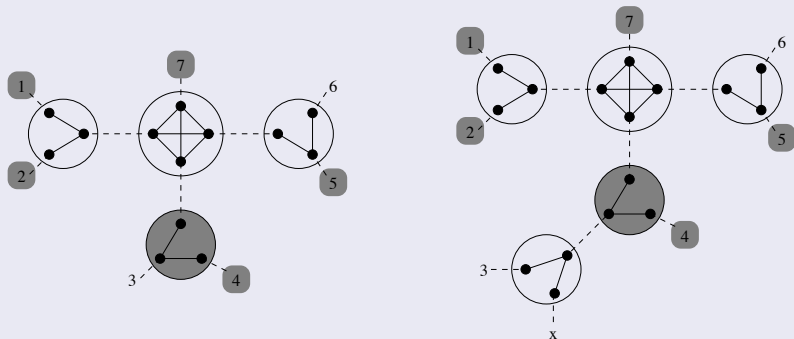
- ① At most one node of $T(S)$ is partially-marked.
- ② Any clique node of $T(S)$ is either fully or partially-marked.
- ③ If there exists a partially-marked node u , then any star node $v \neq u$ of $T(S)$ is oriented towards u if and only if it is fully-marked.
- ④ Otherwise, there exists a tree-edge e of $T(S)$ towards which any star node of $T(S)$ is oriented if and only if it is fully-marked.





The insertion fails: the two singly-marked nodes are oriented towards the partially-marked node !





The insertion succeeds: in $G_S(T, \mathcal{F})$, we have $N(x) = S$

Insertion algorithm

- 1 Extract $T(S)$ (require an arbitrary orientation of $ST(G)$);

Insertion algorithm

- 1 Extract $T(S)$ (require an arbitrary orientation of $ST(G)$);
- 2 Check the mark-type of the nodes and look for an insertion node or edge;

Insertion algorithm

- 1 Extract $T(S)$ (require an arbitrary orientation of $ST(G)$);
- 2 Check the mark-type of the nodes and look for an insertion node or edge;
- 3 Insert the node by either subdividing the insertion edge, or splitting the insertion node, or attaching x to the insertion node.

Cographs

Let (T, \mathcal{F}) be a rooted graph-labelled tree, and S be a subset of leaves of T . A node u of $T(S)$ is:

- **fully-marked** by S if for any children v of u , T_v contains some leaves of S ;

Cographs

Let (T, \mathcal{F}) be a rooted graph-labelled tree, and S be a subset of leaves of T . A node u of $T(S)$ is:

- **fully-marked** by S if for any children v of u , T_v contains some leaves of S ;
- **singly-marked** by S if there is a unique child v such that T_v contains some leaves of S ;

Cographs

Let (T, \mathcal{F}) be a rooted graph-labelled tree, and S be a subset of leaves of T . A node u of $T(S)$ is:

- **fully-marked** by S if for any children v of u , T_v contains some leaves of S ;
- **singly-marked** by S if there is a unique child v such that T_v contains some leaves of S ;
- **partially-marked** otherwise.

Theorem (Cograph incremental characterization [CPS'85])

Let G be a cograph and $MD(G) = (T, \mathcal{F})$ be its modular decomposition tree. Then $G + (x, S)$ is a cograph iff:

- 1 At most one node of $T(S)$ is partially-accessible.

Theorem (Cograph incremental characterization [CPS'85])

Let G be a cograph and $MD(G) = (T, \mathcal{F})$ be its modular decomposition tree. Then $G + (x, S)$ is a cograph iff:

- 1 At most one node of $T(S)$ is partially-accessible.
- 2 Any series node of $T(S)$ is either fully or partially-accessible.

Theorem (Cograph incremental characterization [CPS'85])

Let G be a cograph and $MD(G) = (T, \mathcal{F})$ be its modular decomposition tree. Then $G + (x, S)$ is a cograph iff:

- 1 At most one node of $T(S)$ is partially-accessible.
- 2 Any series node of $T(S)$ is either fully or partially-accessible.
- 3 If there exists a partially-accessible node u , then a parallel node $v \neq u$ of $T(S)$ is a descendant of u if and only if it is fully-accessible.

Theorem (Cograph incremental characterization [CPS'85])

Let G be a cograph and $MD(G) = (T, \mathcal{F})$ be its modular decomposition tree. Then $G + (x, S)$ is a cograph iff:

- ① At most one node of $T(S)$ is partially-accessible.
- ② Any series node of $T(S)$ is either fully or partially-accessible.
- ③ If there exists a partially-accessible node u , then a parallel node $v \neq u$ of $T(S)$ is a descendant of u if and only if it is fully-accessible.
- ④ Otherwise, there exists a tree-edge $e = uw$ of $T(S)$ such that a parallel node $v \neq u$ of $T(S)$ is a descendant of u if and only if it is fully-accessible.

- 1 Definitions and preliminaries
- 2 Vertex modification of totally decomposable graphs
- 3 Conclusion and on-going work

Remarks

- There exists a characterization of cographs in terms of split tree.
- The graph-labelled tree representation of DH graphs yields an intersection model characterizing DH graphs.
- The edge-modification seems also to be possible using the split tree representation.

On-going work

- **Circle graphs and permutation graphs:** properties similar than for prime permutation graphs are observed for prime circle graphs.
- **Computation of the split tree:** An algorithm scheme like Ehrenfeucht et al's one for the modular decomposition can be design for the split decomposition.
- **Factorizing permutation:** such a concept would play the same role as for modular decomposition.
- **Generalization to other decompositions:** apply this graph-labelled tree approach to other decompositions.