# On the (non-)existence of polynomial kernels for $P_l$-free edge modification problems

**Christophe Paul**

CNRS - LIRMM, Montpellier France

Joint work with
**Sylvain Guillemot and Anthony Perez**

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## Parameterized Π-graph edge modification problems

- Given a graph $G = (V, E)$, a parameter $k$
- Find $F \subseteq V^2$ of size at most $k$ such that $H = (V, E \triangle F)$ satisfies Π.

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**Some examples**

- maximal planar subgraph:
  remove at most $k$ edges to get a planar graph
- interval completion:
  add at most $k$ edges to get an interval graph
- cluster editing:
  edit at most $k$ edges to get a disjoint union of cliques

**Most of these problems are NP-complete**

Let $\Pi$ is an hereditary graph property characterized by a finite set of forbidden induced subgrahs.

Theorem [Cai'96]:
The parameterized $\Pi$-graph edge modification problems are FPT

What about polynomial kernels ?

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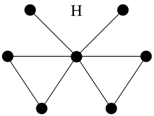
What about polynomial kernels ?

Theorem [Kratsch, Wahlström'09]
The parameterized $H$-free graph edge deletion problem do not have a polynomial kernel unless NP⊆ coNP/Poly



Cubic vertex kernels for cograph (i.e. $P_4$-free graphs)

- Modular decomposition of a graph
- Reduction of the modular decomposition tree
- $P_4$-sunflower rule

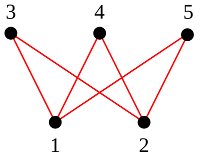
Kernel lower bound for $P_{13}$-free graphs (and $C_{12}$-free graphs)

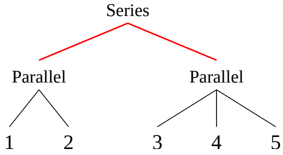
- A OR-compositional toy problem
- A polynomial parameter transformation

# Cograph edge deletion kernel

$G$ is a cograph if it can be built from the single vertex graph by:

- [disjoint / parallel union] $G = G_1 \oplus G_2$
- [series composition] $G = G_1 \otimes G_2$ (add all possible edges between $G_1$ and $G_2$)



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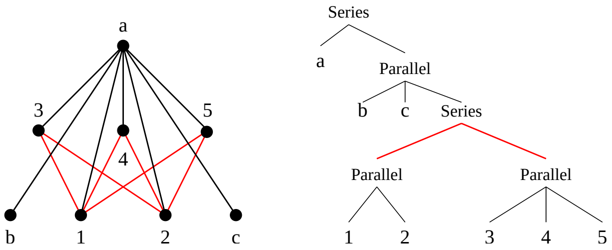

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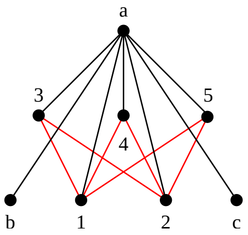


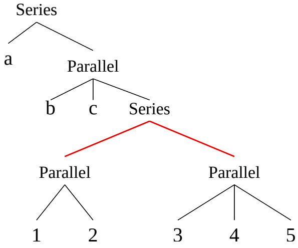


Theorem [Folklore]:
Every cograph has a canonical tree-decomposition (called cotree)

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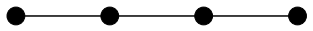
- the $P_4$-free graphs 
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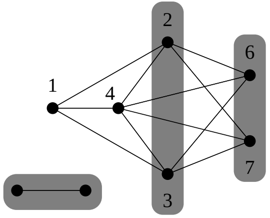
A subset of vertices $M$ of a graph $G = (V, E)$ is a module if

$$\forall x \in V \setminus M, \text{ either } M \subseteq N(x) \text{ or } M \cap N(x) = \emptyset$$




Examples of modules

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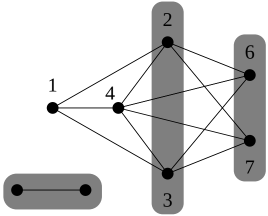
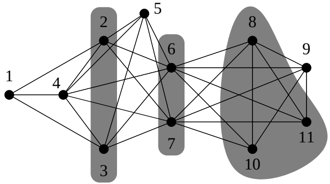


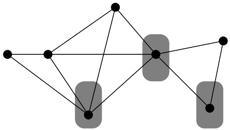

Examples of modules

- connected components
- connected components of $\overline{G}$
- any vertex subset of the complete graph (or the stable)

# The modular decomposition tree (1)

A partition $\mathcal{P} = \{M_1, \ldots M_r\}$ of the vertex set of a graph $G$ is a modular partition of $G$ if any part $M_i \in \mathcal{P}$ is a module of $G$.



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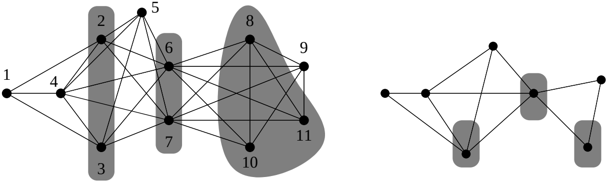


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A graph is prime if all its modules are trivial: e.g. the $P_4$.

# The modular decomposition tree (2)

Theorem [Gal'67,CHM81] Let $G$ be a graph. Then either

1. $G$ is not connected (parallel), or
2. $\overline{G}$ is not connected (series), or
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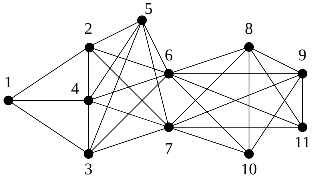
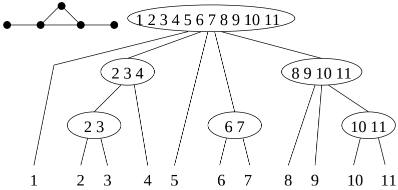
A module is strong if it does not overlap any other module



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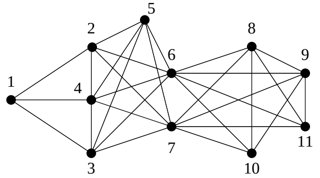


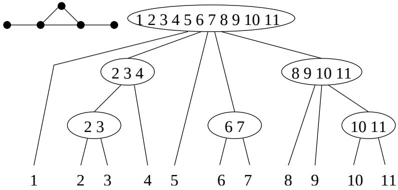


MD-tree can be computed in $O(n+m)$

Cubic vertex kernels for cograph (i.e. $P_4$-free graphs)

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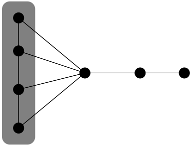
Kernel lower bound for $P_{13}$-free graphs (and $C_{12}$-free graphs)

- A OR-compositional toy problem
- A polynomial parameter transformation

Observation: If $M$ is a module and $abcd$ is a $P_4$, then either $\{a, b, c, d\} \subseteq M$ or $|\{a, b, c, d\} \cap M| = 1$



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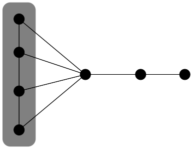

- work independently on quotient graphs and subgraphs induced by modules

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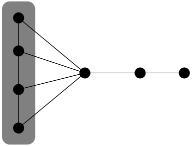

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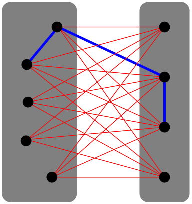
Rule 1 Remove the $P_4$-free connected components

Rule 2 If $C = G_1 \otimes G_2$ is a connected component of $G$, then replace

$$C \text{ by } G_1 \oplus G_2$$

(i.e. remove the edges between $G_1$ and $G_2$)



Lemma: Let $M$ be a non-trivial module of the input graph $G$. Let $F_{opt}$ be optimal for $G$ and $F_M$ be optimal for $G[M]$. Then

$$\boxed{F = (F_{opt} \setminus F_{opt}[M]) \cup F_M \text{ is optimal for } G}$$

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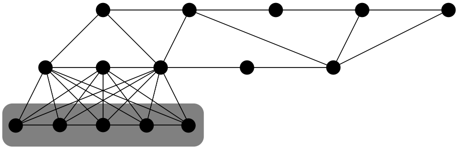
Reduction rule 3

Let $M$ be a non-trivial module of some connected component. If $M$ is not an independent set of size at most $k+1$, then



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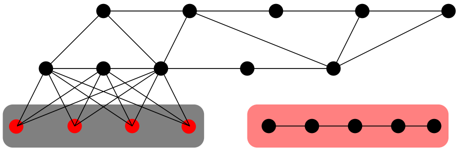

$$(G, k) \equiv (G' \oplus G[M], k)$$

where $G'$ is obtained from $G$ by substituting $M$ with an independent set of size $\min\{|M|, k+1\}$

- Neither the size of the graph nor the parameter decrease.
- There is an optimal solution preserving the modules of $G$.

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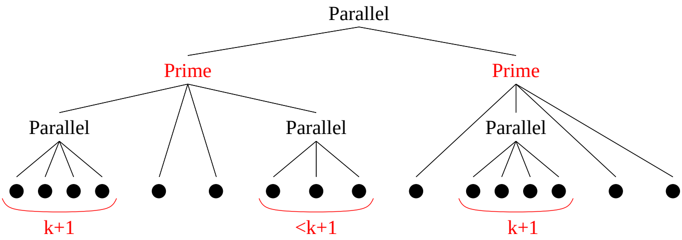
**Lemma**

If $G$ is reduced under Rules 1, 2 and 3, then the modular-tree decomposition of every connected component of $G$ satisfies

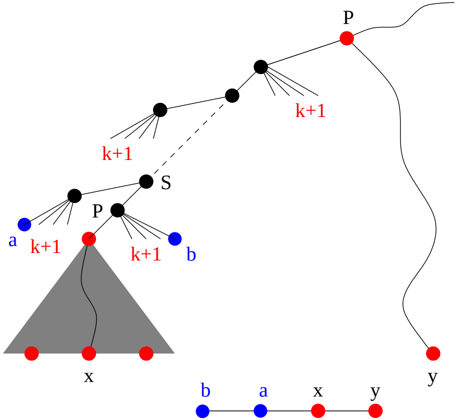
- the root is prime
- every non-root node is parallel with at most $k+1$ leaves
- there is no other node



Sunflower rule: if there are at least $k+1$ edges pairwise intersecting only on $e$, remove $e$ and decrease the parameter.

Theorem [Guillemot, P., Perez]:

Cograph edge deletion has a cubic vertex kernel



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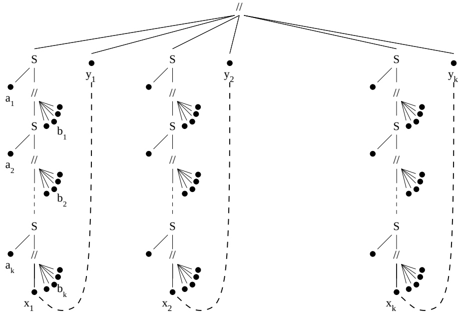


The bound is tight for these rules

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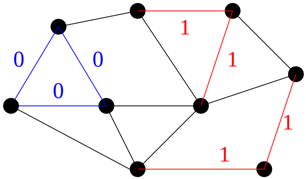
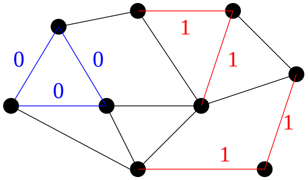



### NOT-1-IN-3-EDGE-TRIANGLE

- A graph $G = (V, E)$, an edge bicolouring $\Phi : E \to \{0, 1\}$ and a parameter $k$
- Can we extend $B$ to a valid edge bicolouring $\Phi'$ of weight at most $k$ ?

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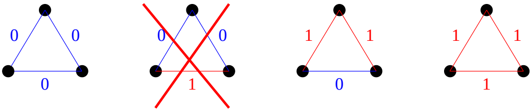
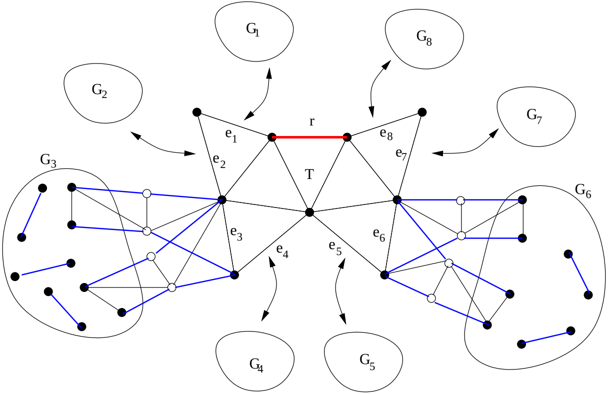


## Lemma

NOT-1-IN-3-EDGE-TRIANGLE is NP-complete and OR-compositional



NOT-1-IN-3-EDGE-TRIANGLE in general graphs

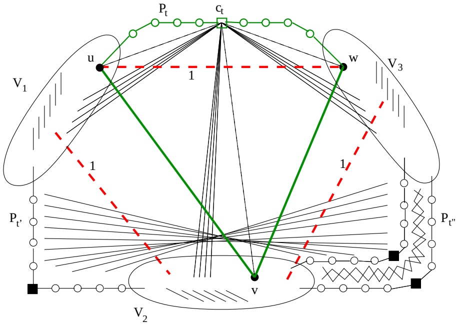
↓

(Polynomial Parameter Transformation - PPT)

↓

NOT-1-IN-3-EDGE-TRIANGLE in tripartite graphs

↓




↓

$P_{l+1}$-free / $C_l$-free edge deletion problem ($l \geqslant 12$)

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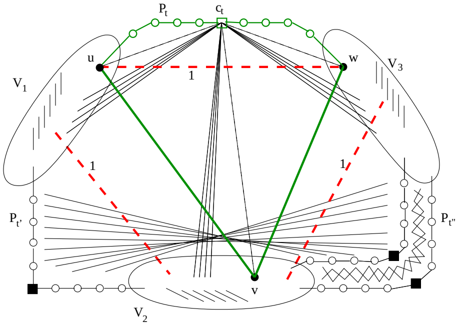


Lemma: A set $C$ of vertices induces a cycle of length $l$ iff $G$ contains a triangle $uvw$ with a unique 1-edge $uw$ and $C = \{u, v, w\} \cup P_t$.

## A related open problem

Given a graph $G = (V, E)$ and an integer $k$, does there exist $F \subseteq V \times V$ such that

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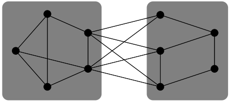
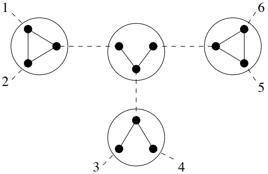
Theorem Let $G$ be a graph. The following are equivalent

- $G$ is a distance hereditary graph
- $G$ is totally decomposable by the split decomposition



A split




the split tree of a DH graph

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Theorem Let $G$ be a graph. The following are equivalent

- $G$ is a distance hereditary graph
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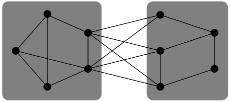
- $G$ has rankwidth 1

  (BUT no finite set of forbidden subgraphs)



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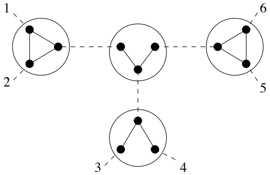


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