Kernelization techniques, Feedback Arc Set in Tournaments and Modular Decomposition

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Joint work with

S. Bessy, F. Fomin, S. Gaspers, A. Perez, S. Saurabh, S. Thomassé

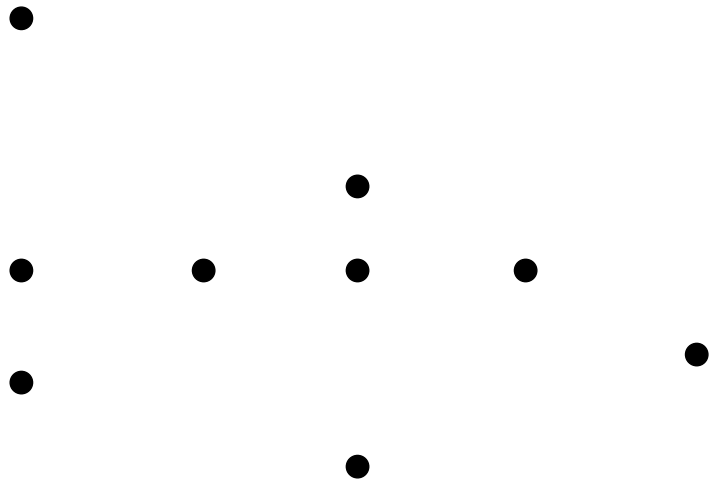
Introduction to kernelization and lower bound techniques

A linear kernel for Feedback Arc Set in Tournament (FAST)

Other modular decomposition based kernels

A FIRST EXAMPLE

Can we cover a set of n points in the plane by at most k lines ?



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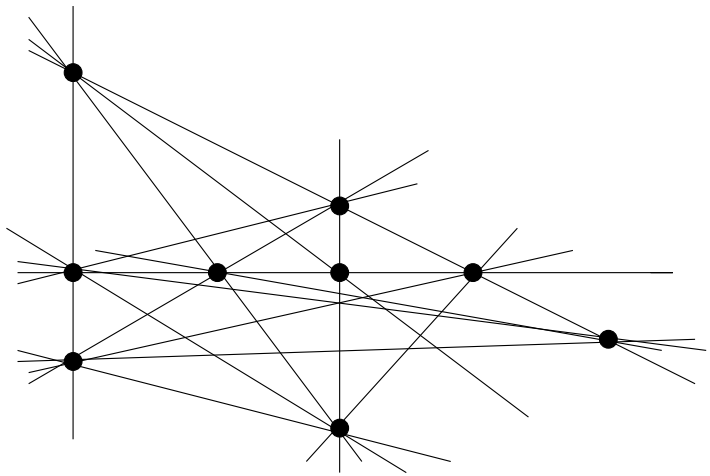

Observation

We can restrict the set of lines to those that are generated by the set of pair of points of S.

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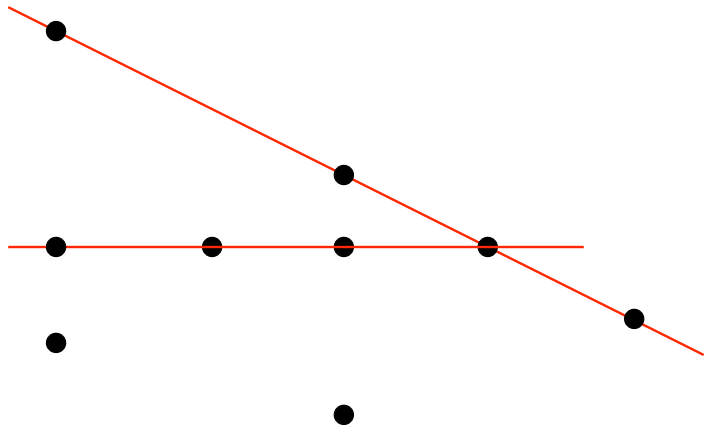

Reduction rule [Sunflower]

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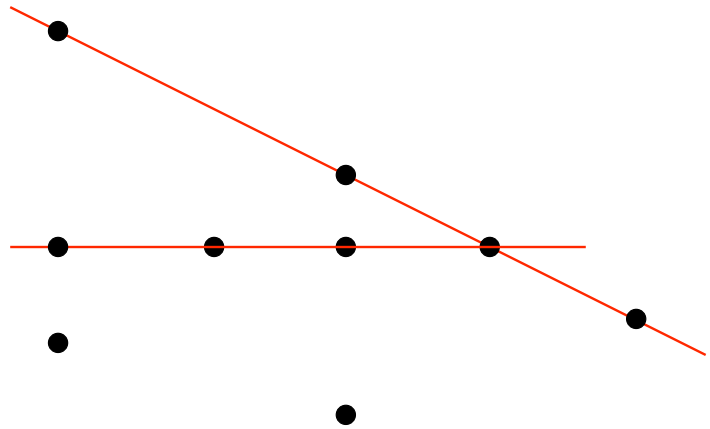

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Théorème

Any instance of the point cover problem can be reduced in polynomial time to an equivalent instance of size at most k^2 points.

KERNELIZATION

A kernelization of a parameterized problem (Q, κ) is a polytime algorithm that transforms

$$\text{an instance } (I, k) \quad \longrightarrow \quad \text{an reduced instance } (I', k')$$

such that

- (I, k) is a YES-instance $\iff$ (I', k') is a YES-instance
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$\Rightarrow$
- if $g(k) \leqslant n$, then solve in time $f(k).n^{O(1)} \leqslant n^{O(1)}$
- if $n < g(k)$, then (I, k) is a kernel.

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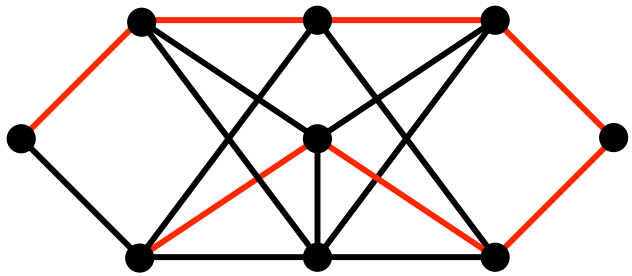
BUT this gives only exponential size kernel !

Does every parameterized problem have a polynomial kernel ?

LONGEST PATH

- A graph $G = (V, E)$, a parameter $k \in \mathbb{N}$
- Does G contain a path of length k ?



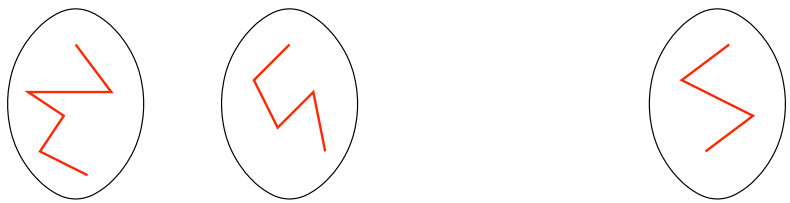
Longest Path is NP-Complete (cf. hamiltonian path) but is FPT by Color Coding [Alon, Yuster and Zwick])

LONGEST PATH

Hypothesis

LONGEST PATH has a poly-size kernelization $\mathcal{A}$

- consider an instance (G, k) built from a series of t instances
  $(G, k) = (G_1, k) \oplus (G_2, k) \oplus \ \ldots \ \oplus (G_t, k)$



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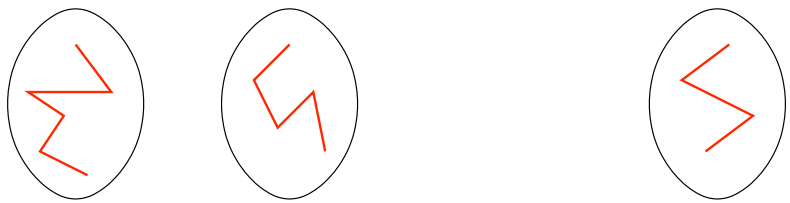

Observation : (G, k) is a YES instance iff $\exists i$ such (G_i, k) is a YES instance.

Question / intuition :

Is it possible that a polytime algorithm $\mathcal{A}$ computes an equivalent instance of size $k^{O(1)}$ (a value possibly much smaller than t) ?

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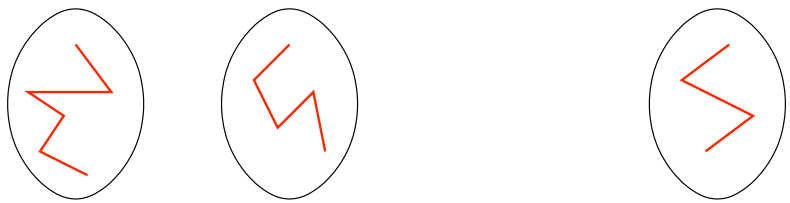

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Is it possible that a polytime algorithm $\mathcal{A}$ identifies which G_i is a YES instance (if any) ?

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Theorem [Fortnow et Santhanam]

The OR-distillation conjecture holds, unless $PH = \Sigma_p^3$

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Observation : The existence of a poly kernel for Longest Path would imply the existence of an OR-compositional algorithm.

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Corollary

Unless $PH = \Sigma_p^3$, LONGEST PATH does not have a polynomial kernel.

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Input: A tournament $T = (V, A)$ and a parameter k

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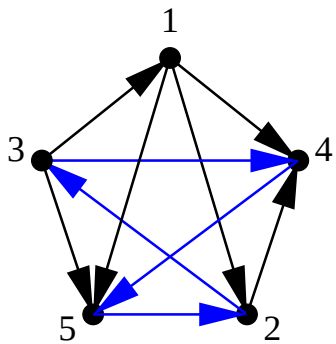

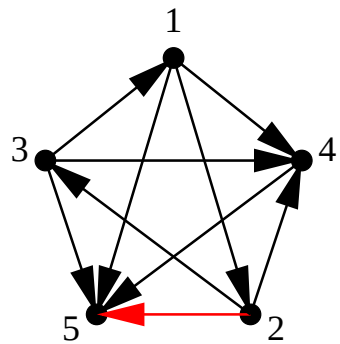

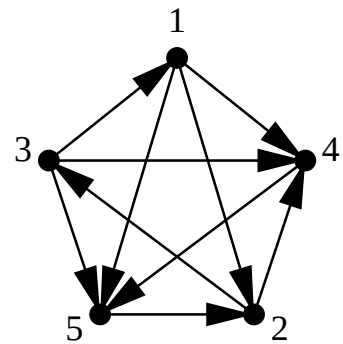

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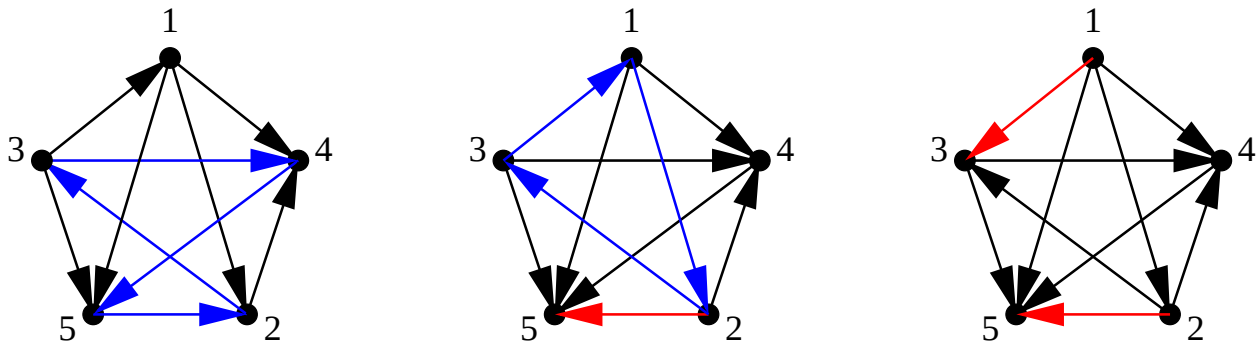

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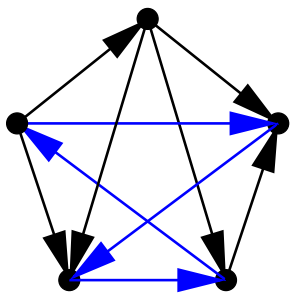

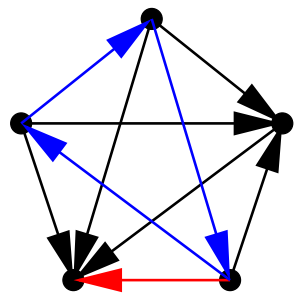

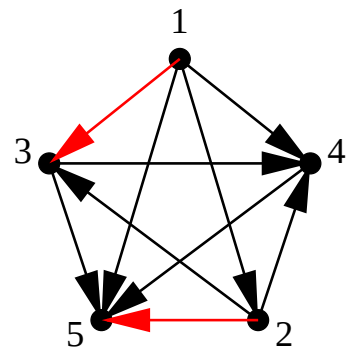

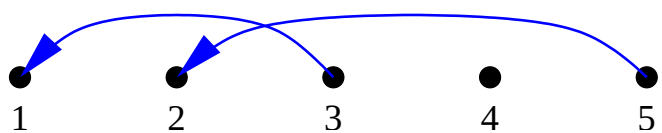

$\longrightarrow$ find a vertex ordering π with at most k backward edges

KNOWN RESULTS

- NP-Complete [Alon'06] [Charbit et al.'07]
- FTP [Raman, Saurabh'06] [Alon et al.'09]
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Our result

k-FAST has a $O(k)$ vertex kernel

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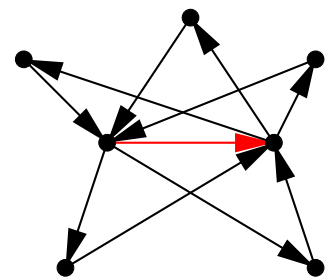

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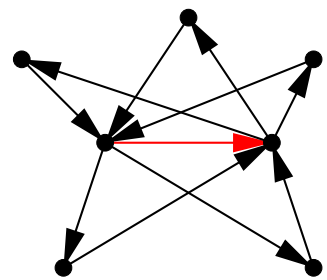

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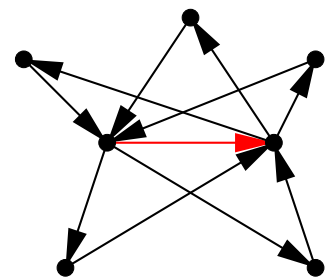
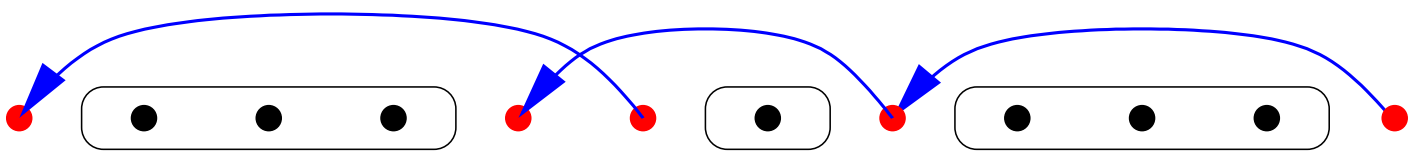
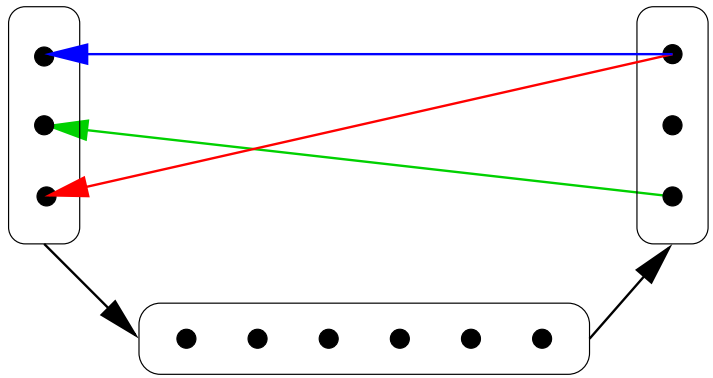


The span $s(\overrightarrow{uv})$ of a backward arc of a reduced tournament is $\leqslant 2k+2$



Rule 3 [acyclic module] Let M be a maximal acyclic module. If there are at most $p = |M|$ arcs from $N^+(M)$ to $N^-(M)$, then reverse all these arcs and decrease k by p.



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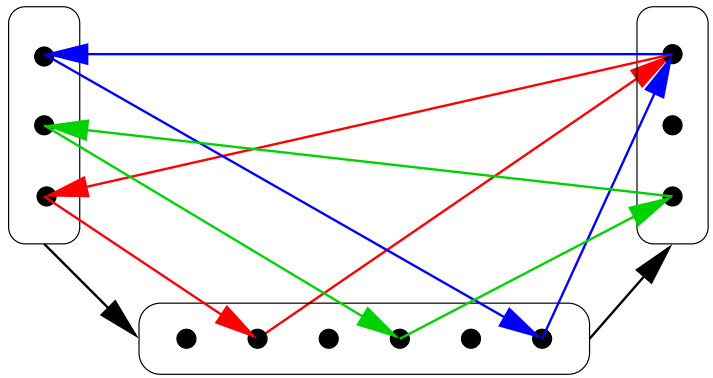

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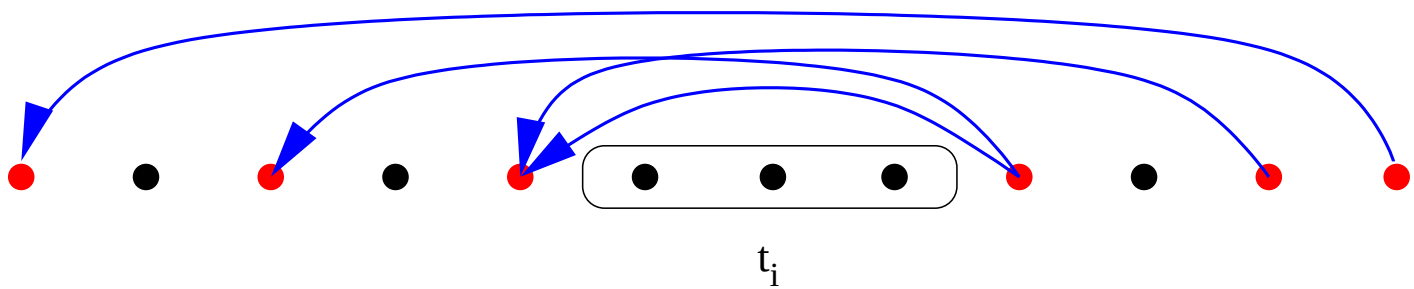
acyclic module Rule $\Rightarrow$

$$\sum t_i^2 \leqslant \sum s(\overrightarrow{uv})$$



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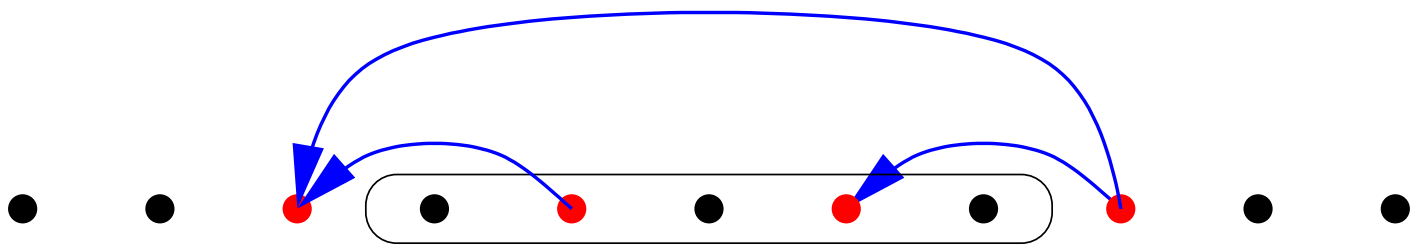
$$\sum t_i^2 \leqslant \sum s(\overrightarrow{uv}) \leqslant k(2k+2)$$



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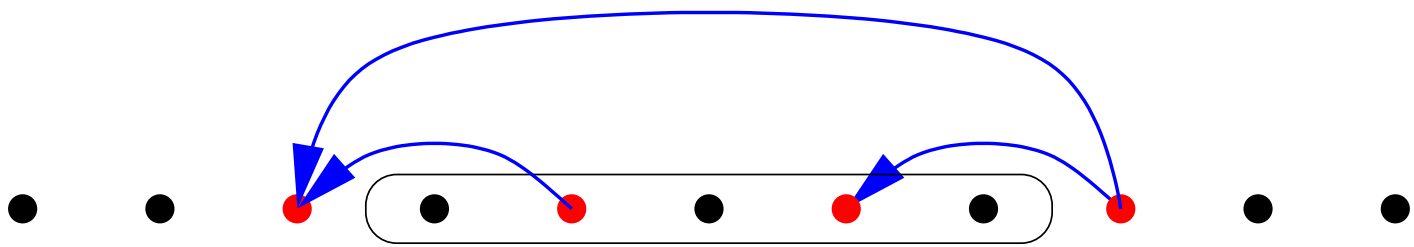
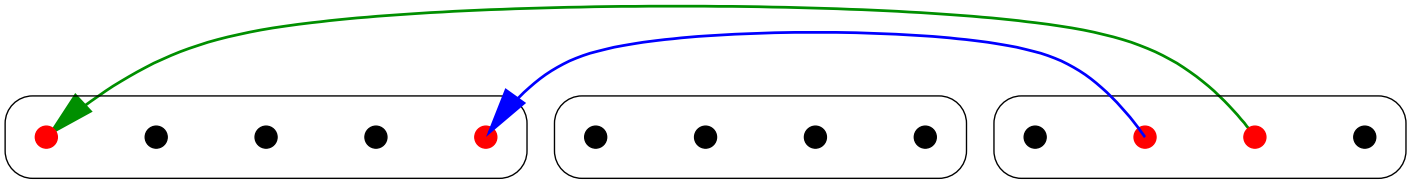


Theorem:

k-FAST has a vertex kernel of size $2k + \sum t_i = O(k\sqrt{k})$.

ACYCLIC MODULE RULE REVISTED



$$\overrightarrow{A} = \overrightarrow{E}\ (\text{external arcs}) \uplus \overrightarrow{L}(\ \text{local arcs})$$

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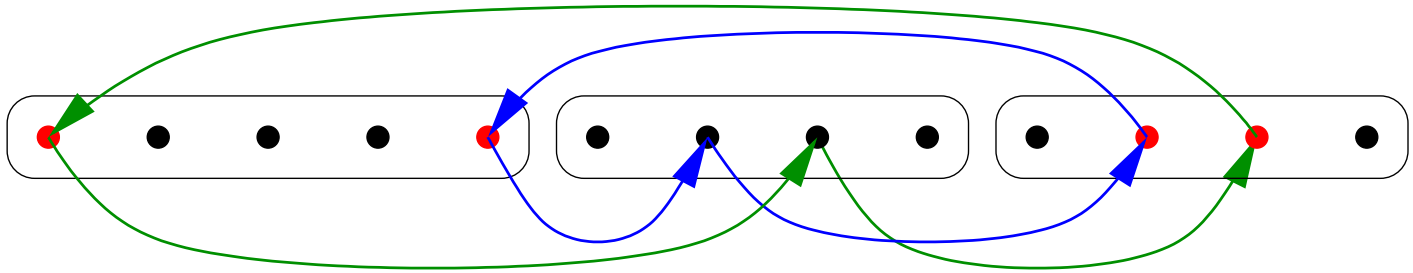

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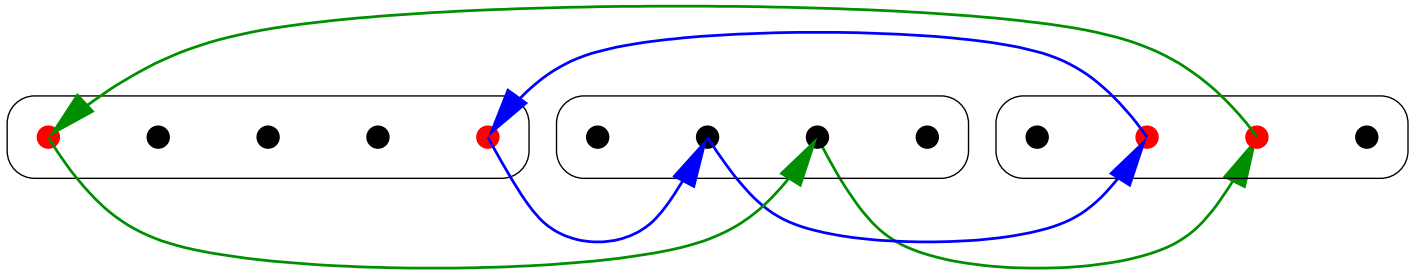
ACYCLIC MODULE RULE REVISTED



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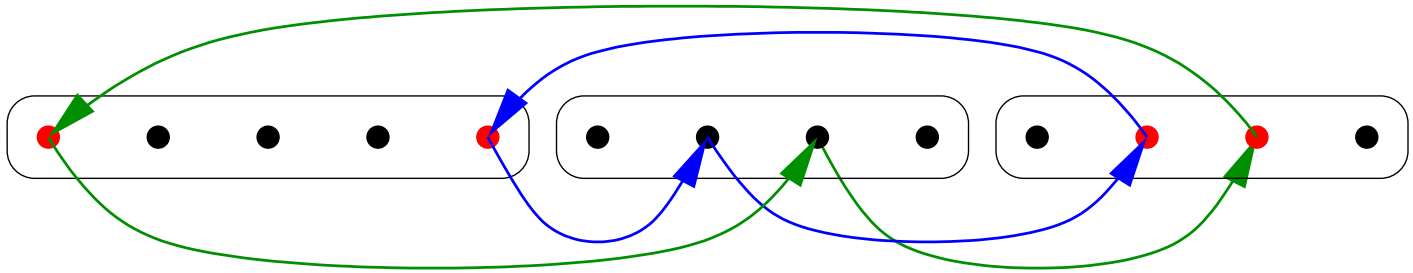

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Rule 3 [safe partition] If $\mathcal{P}$ is a safe partition
then reverse *backwards*$(\overrightarrow{E})$ and decrease k by $|$*backwards*$(\overrightarrow{E})|$.

GOAL

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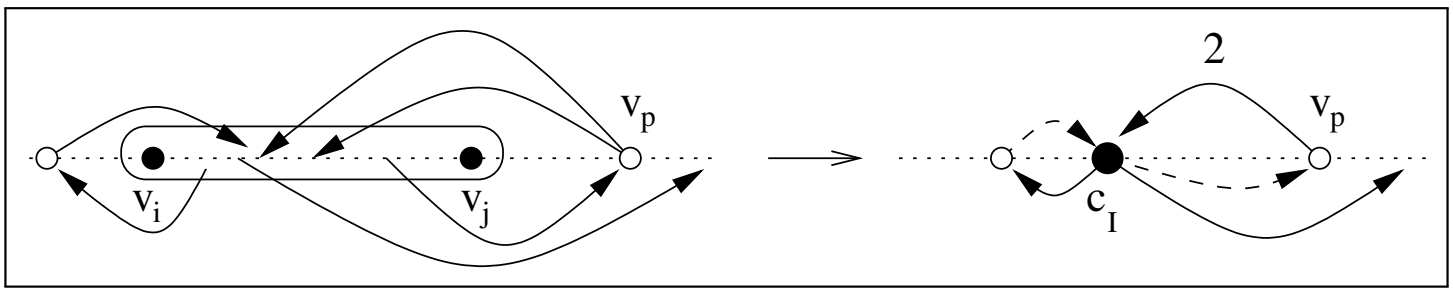
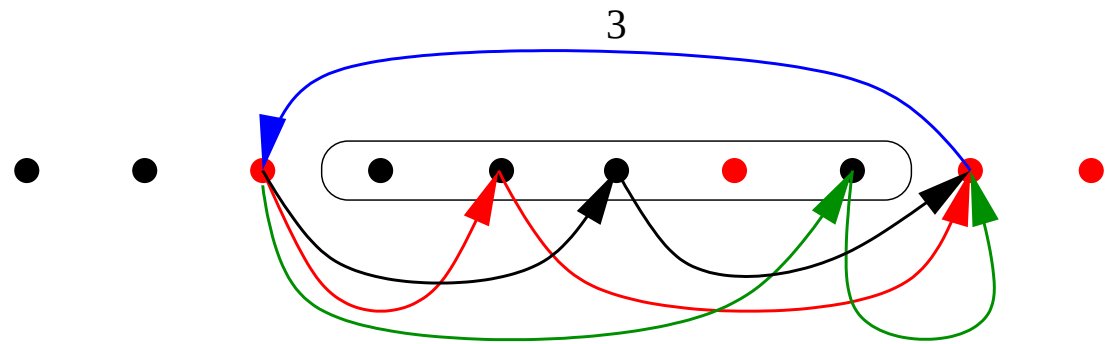
Interval contraction



CERTIFICATE

A certificate $c(\overrightarrow{uv})$ for a backward arc $\overrightarrow{uv}$ of weight ω is a collection of ω arc-disjoint forward paths in the span of $\overrightarrow{uv}$



If $\overrightarrow{B}$ is a set of backward-arcs, then

$$c(\overrightarrow{B}) = \{c(\overrightarrow{uv}) \text{ pairwise disjoint } \mid \overrightarrow{uv} \in \overrightarrow{B}\}$$

Lemma: If T is an ordered backward weighted tournament st.

$$\forall\ I = [v_i \dots v_j] \ : \quad 2\ .\ \omega(I) \leqslant |I| - 1$$

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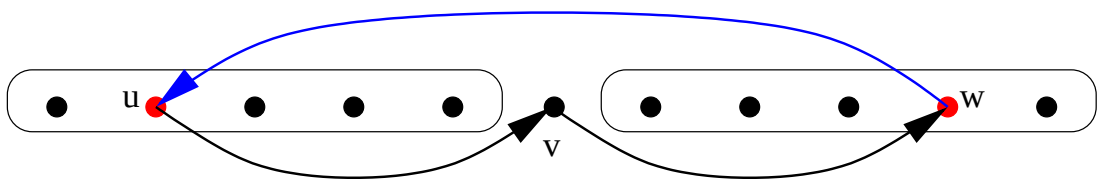

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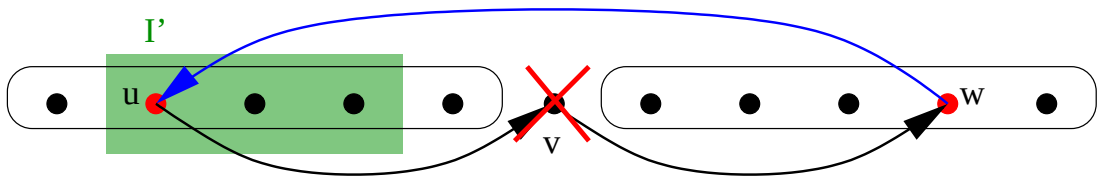

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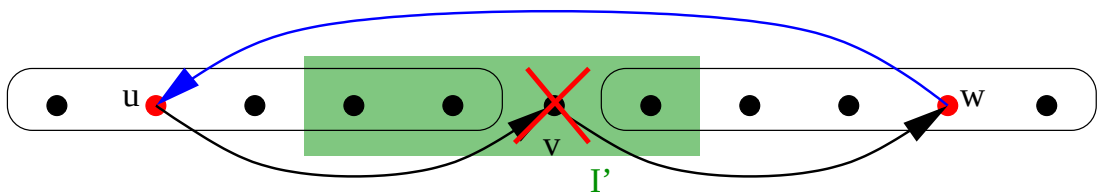

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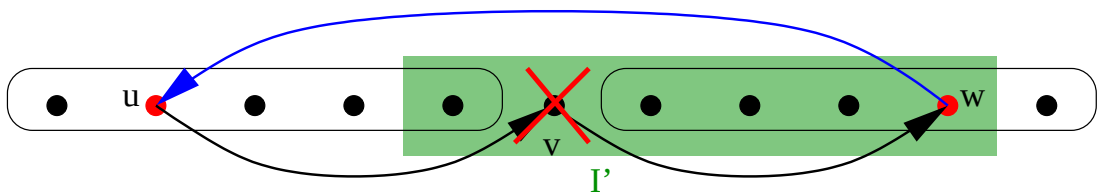

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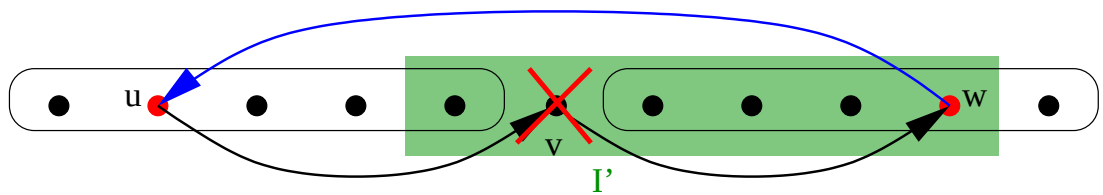

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By its choice, $\overrightarrow{wu}$ cannot be used once reversed

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1. $[v_1 \dots v_n]$ is the only critical interval : $2\ .\ \omega(I) = |I| - 1$
2. if there exists a critical interval $I \neq V$,
   then $T[I]$ and T/I can be disjointly certified

Lemma: If T is an ordered backward weighted tournament st.

$$|V(T)| \geqslant 2p+1 \quad \text{and} \quad |\overrightarrow{B} = \{\text{backward-arcs of } T\}| \leqslant p$$

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- Let I be an interval st $2 \,.\, \omega(I) > |I| - 1$ then

contract I into v_I, find a safe partition $\mathcal{P}$ of T/I
$\mathcal{P}'$ obtained by substituting v_I by I is a safe partition of T

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- if $|V(T)| \leqslant (2+\epsilon).k$, then apply the safe partition rule and then the other rules
- when it stops, either we have the kernel or $k = 0$

Introduction to kernelization and lower bound techniques

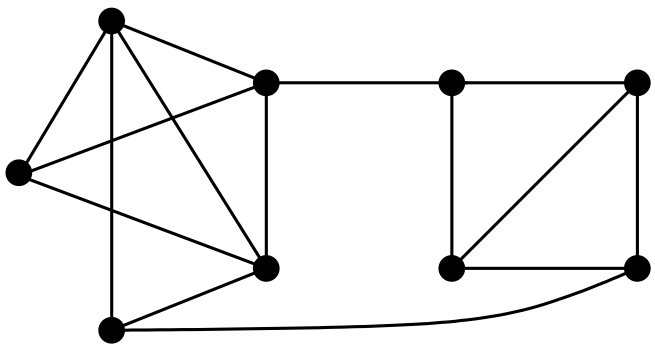
A linear kernel for Feedback Arc Set in Tournament (FAST)

Other modular decomposition based kernels

CLUSTER EDITING

Given a graph $G = (V, E)$, find $F \subseteq V \times V$ such that

- $|F| \leqslant k$ et
- $H = (V, E \triangle F)$ is the disjoint union of cliques



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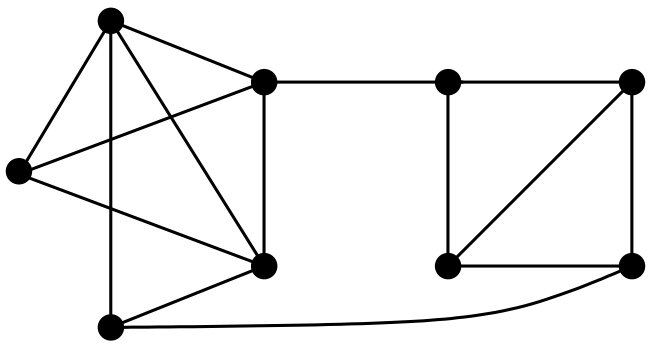

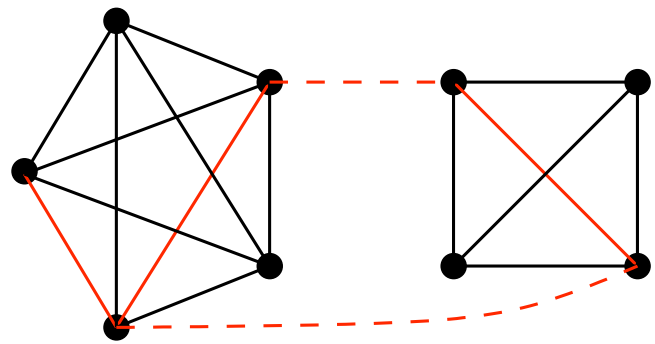

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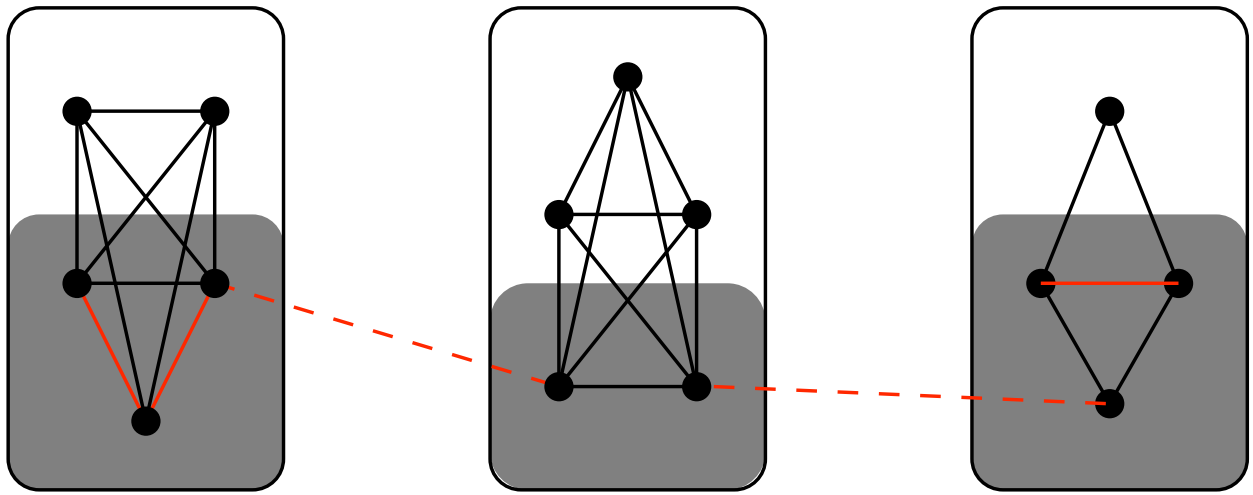

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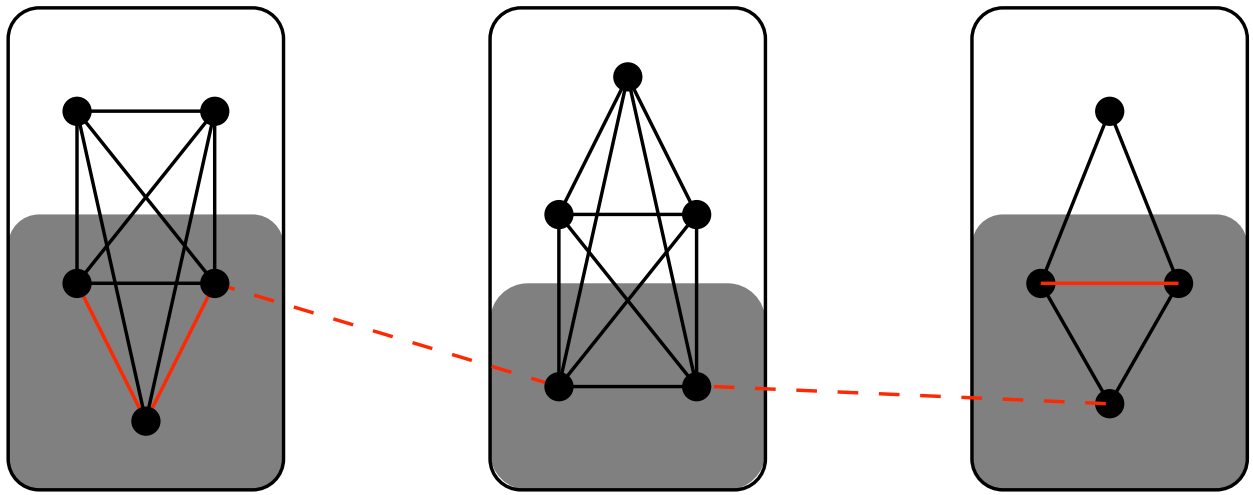

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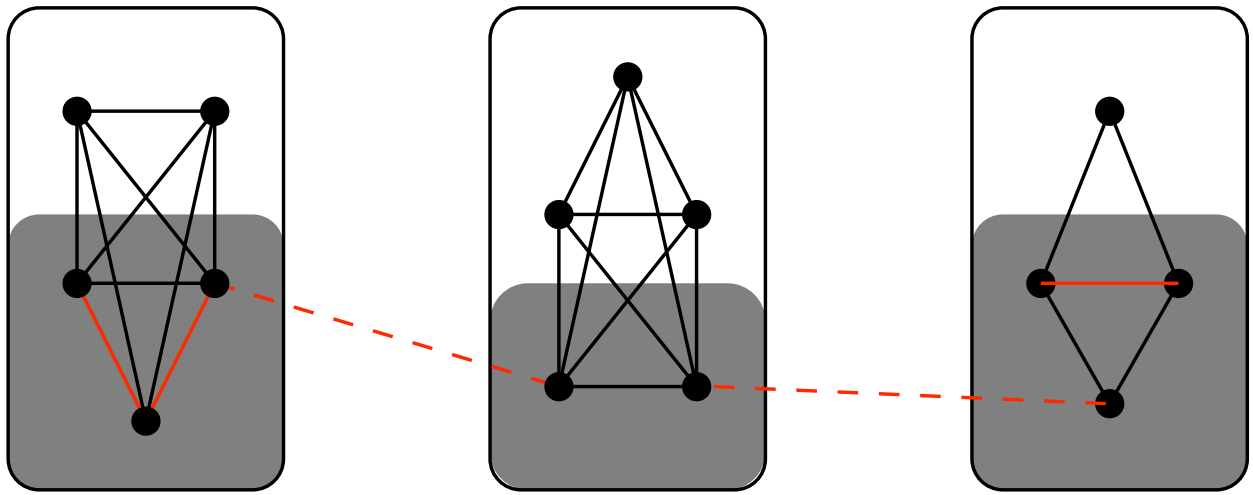

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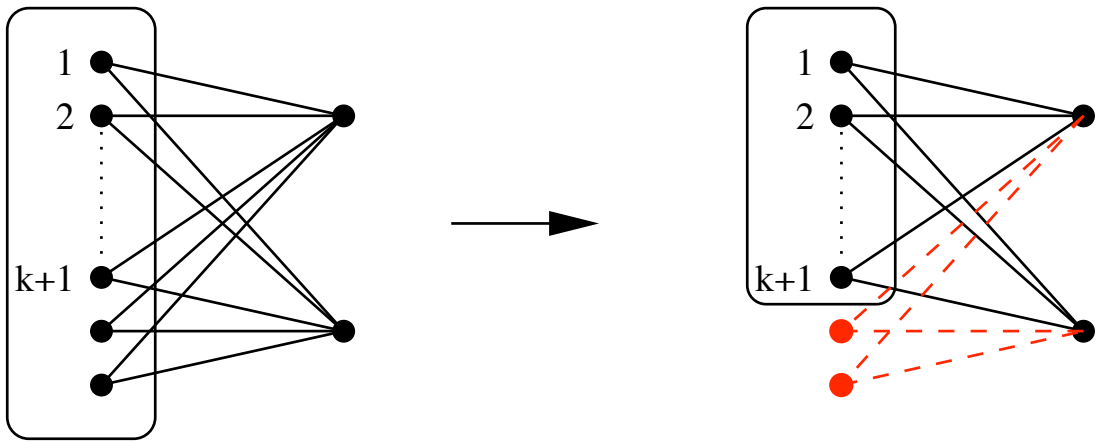

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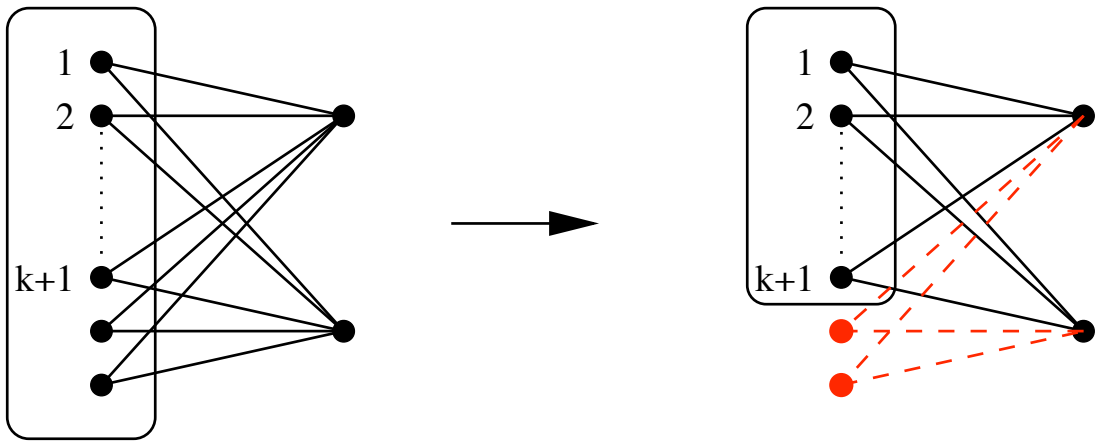

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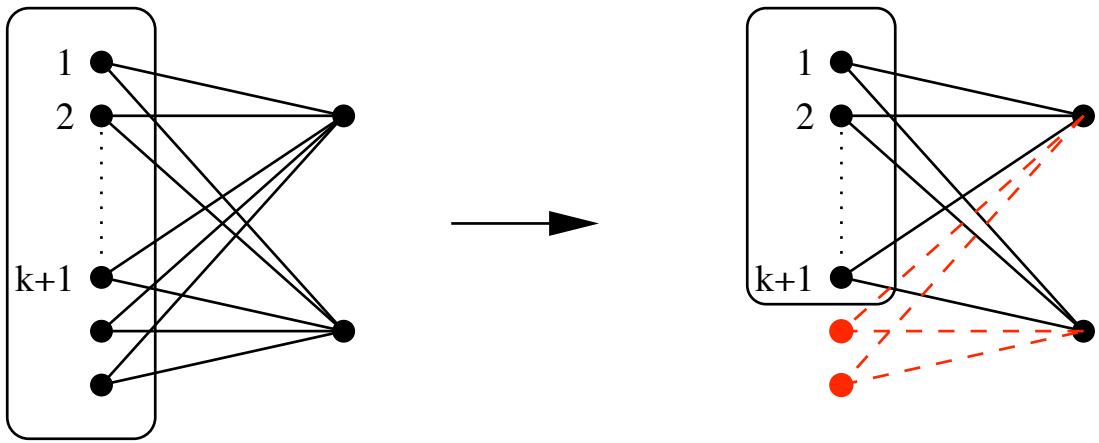

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$\longrightarrow$ a $4k$ vertex kernel is known [Guo'07, Fellows et al.'07]

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- Closest 3-Leaf Power
- Flip Consensus Tree
- Cograph Editing

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THANK YOU !