Algorithmic Aspects of Modular Decomposition

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Habilitation (in French) avalaible at `www.lirmm.fr/~paul`

Joint work with

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- A. Bretscher, D. Corneil (U. of Toronto),
- M. Habib, F. de Montgolfier (LIAFA),
- L. Viennot (INRIA Rocquencourt)

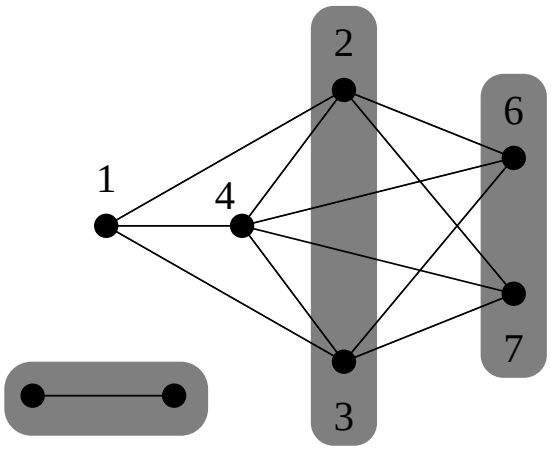
Some motivations

- A very natural operation on discrete structures (used is the proof of the perfect graph theorem)
- Help to capture the structure of many graphs families (basis of a theory for comparability graphs)
- Divide and conquer paradigm for optimization problems
- Provide compact representation of graphs (even for dynamic graphs)
- ...

1. Definitions and preliminaries
2. Ehrenfeucht et al.'s algorithm
   - Principle
   - Computation of $\mathcal{M}(G, v)$
   - Computation of $MD(G_{/\mathcal{M}(G,v)})$
3. New algorithmic trends
   - Factoring permutation
   - LexBFS

Modules

A subset of vertices M of a graph $G = (V, E)$ is a **module** iff
$\forall x \in V \setminus M$, either $M \subseteq N(x)$ or $M \cap N(x) = \emptyset$



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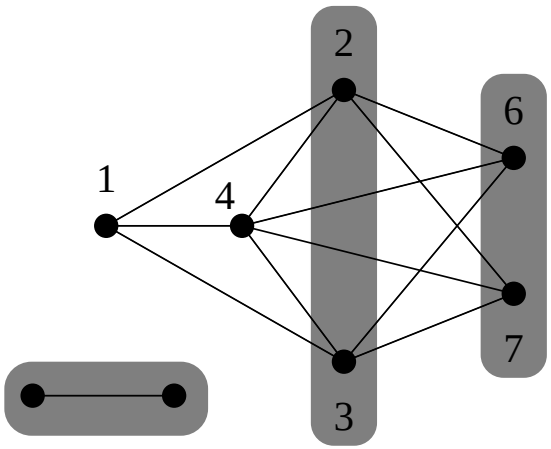

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- connected components of $\overline{G}$

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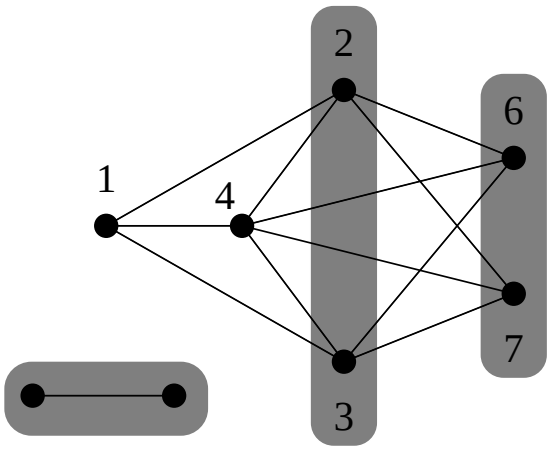
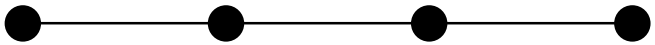


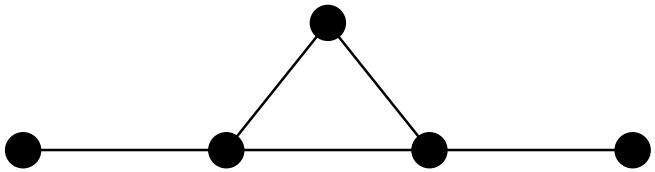
Examples of modules

- connected components
- connected components of $\overline{G}$
- any vertex subset of the complete graph (or the stable)

Prime graphs and degenerate graphs

- A graph is **prime** is all its modules are trivial: e.g. the P_4.
- A graph is **degenerate** if any subset of vertices is a module: cliques and stables.





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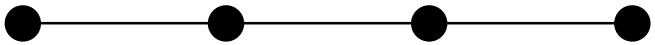
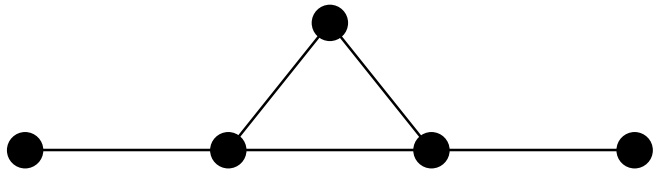


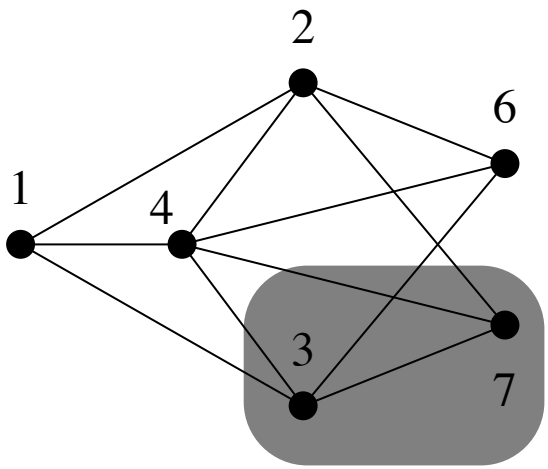


Lemma (Cournier, Ille 91)

If G is a prime graph, then any vertex but at most one belongs to a P_4.

Splitter

For a graph $G = (V, E)$, the vertex x is a **splitter** for a set S of vertices if $\exists y, z \in S$ with $xy \in S$ and $xz \notin E$. We say that x **separate** y and z.



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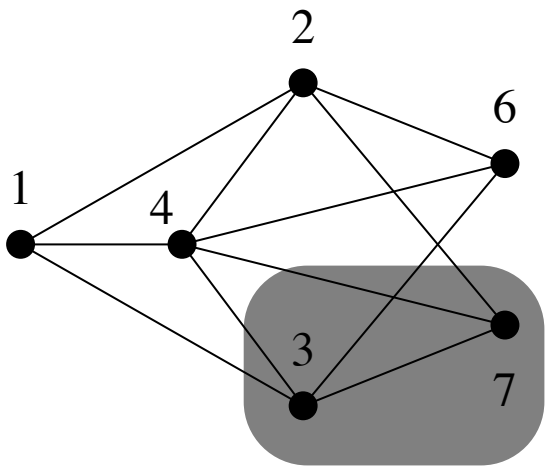
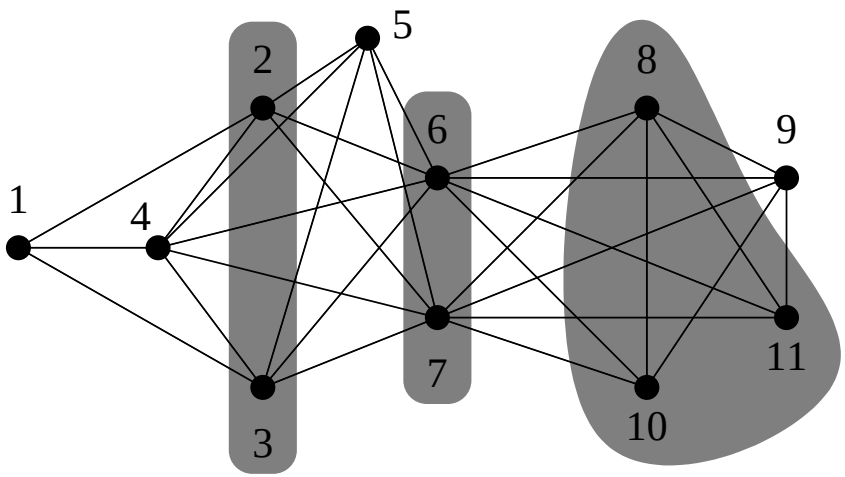


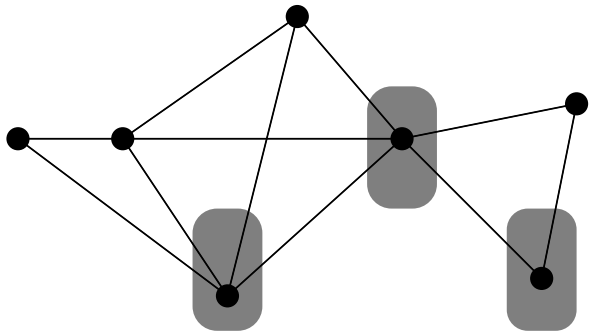
Lemma

If x is a splitter for the set S, then any module M containing S must also contain x.

Modular partition

A partition $\mathcal{P}$ of the vertex set of a graph $G = (V, E)$ is a **modular partition** of G if any part is a module of G.





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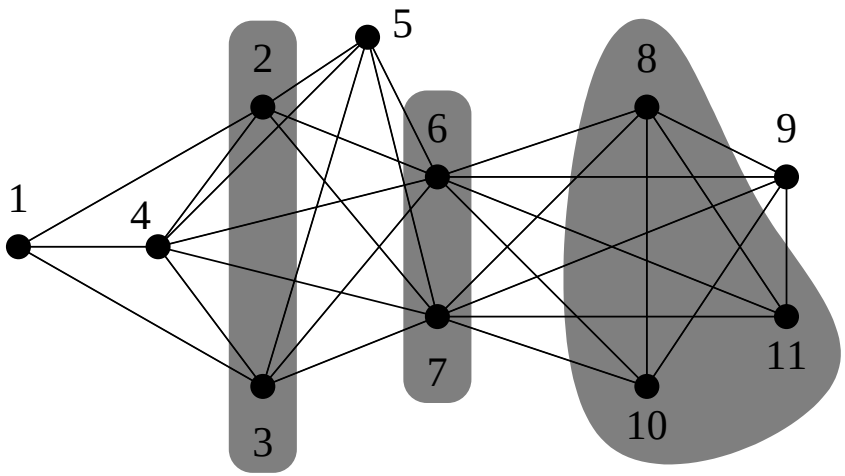
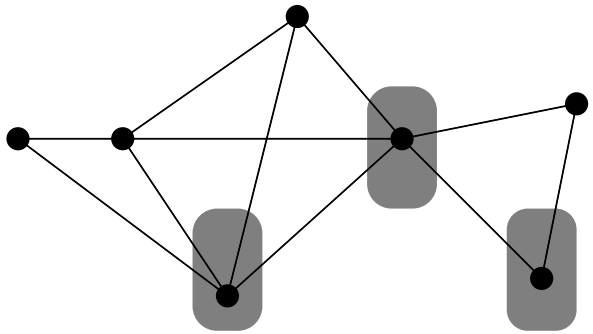
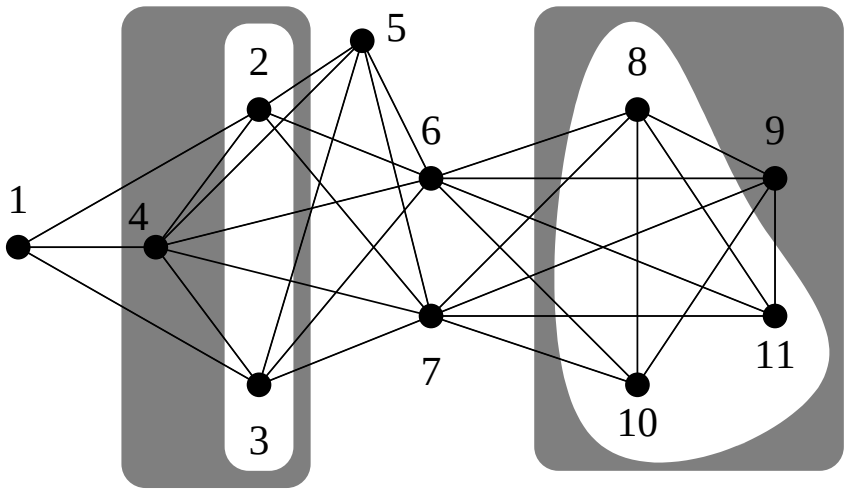


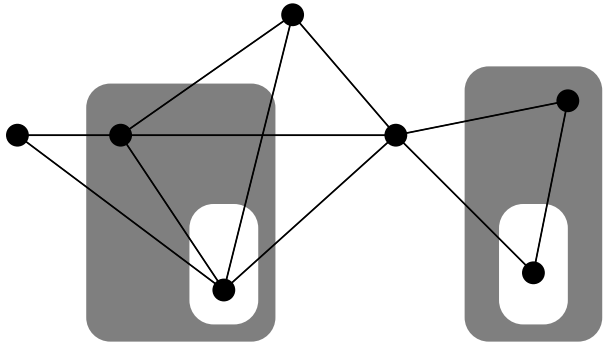


Let $\mathcal{P}$ be a modular partition of a graph $G = (V, E)$. The **quotient graph** $G_{/\mathcal{P}}$ is the induced subgraph obtained by choosing one vertex per part of $\mathcal{P}$.

Lemma (MR84)

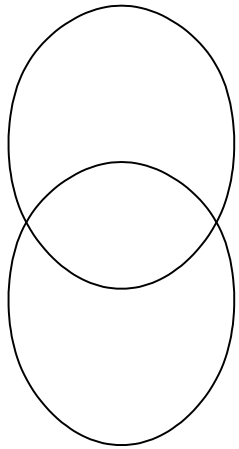
Let $\mathcal{P}$ be a modular partition of $G = (V, E)$.
$\mathcal{X} \subseteq \mathcal{P}$ is a module of $G_{/\mathcal{P}}$ iff $\cup_{M \in \mathcal{X}} M$ is a module of G.





Lemma

If M and M' are two overlapping modules then



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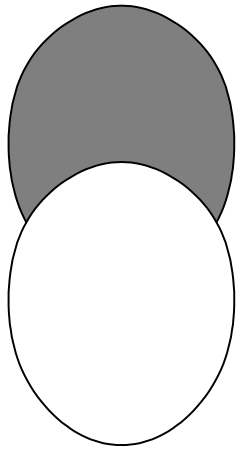

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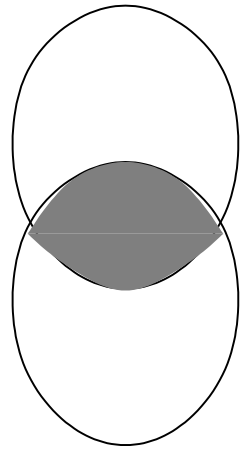

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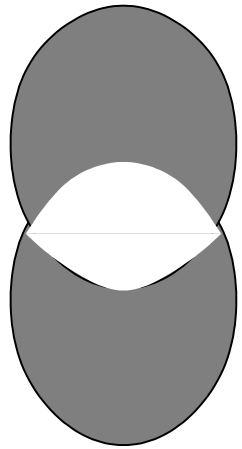
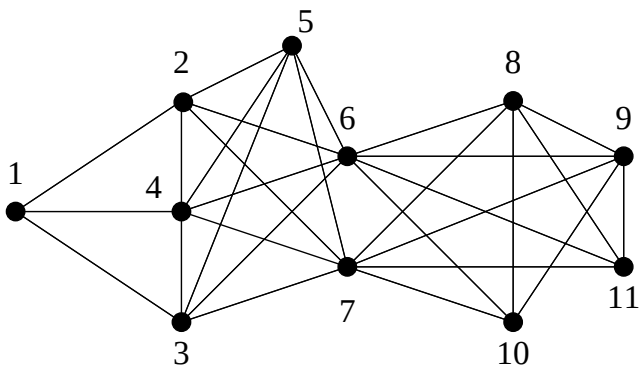
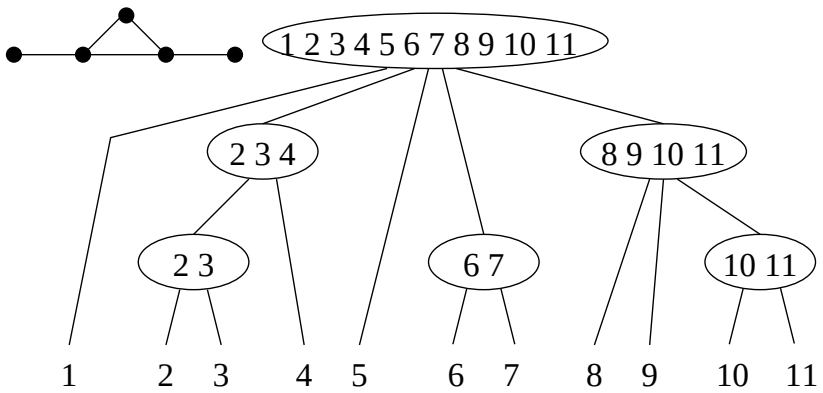


- The set of modules of a graph defines a **partitive family**
- A module is **strong** if it does not overlap any other module

The set of strong modules is nested into an inclusion tree (called the **modular decomposition tree** $MD(G)$ of G).





Theorem (Gal'67,CHM81)

Let $G = (V, E)$ be a graph. Then either

1. *G is not connected, or*
2. *$\overline{G}$ is not connected, or*
3. *$G_{/\mathcal{M}(G)}$ is a prime graph, with $\mathcal{M}(G)$ the modular partition containing the maximal strong modules of G.*

$\Rightarrow$ "Brut force" algorithm in $O(n^3)$ time

Some algorithms

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That was only for undirected graphs!!!

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- Principle
- Computation of $\mathcal{M}(G, v)$
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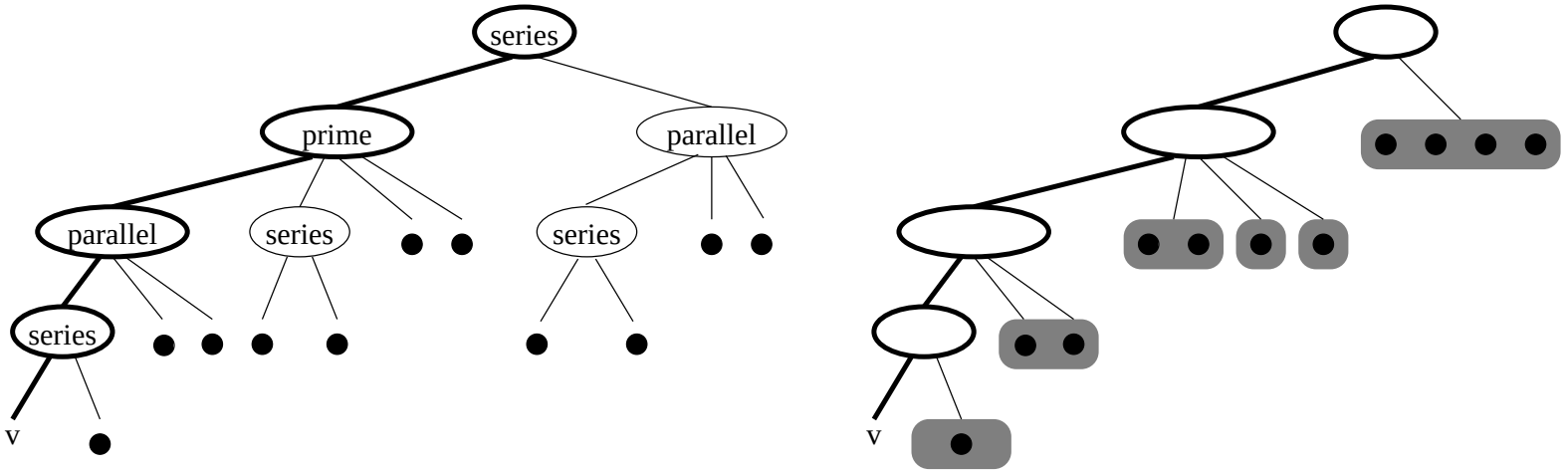
$\mathcal{M}(G, v)$ is composed by $\{v\}$ and the maximal modules of G not containing v.



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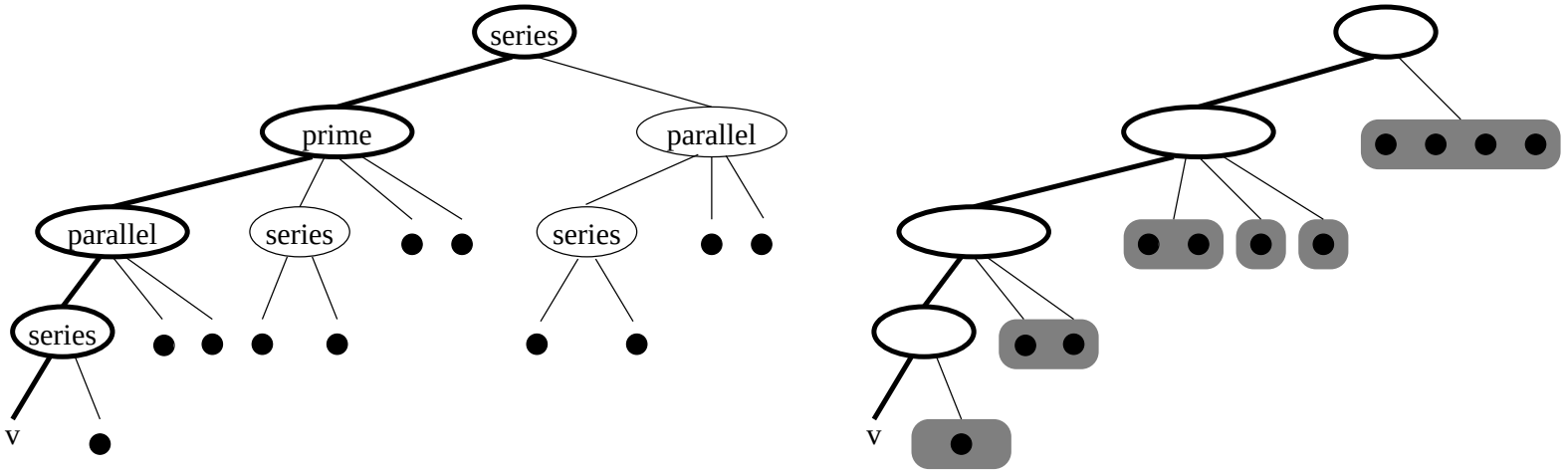

Principle of the Ehrenfeucht et al.'s algorithm

1. Compute $\mathcal{M}(G, v)$
2. Compute $MD(G_{/\mathcal{M}(G,v)})$
3. For each $\mathcal{X} \in \mathcal{M}(G, v)$ compute $MD(G[\mathcal{X}])$

Lemma

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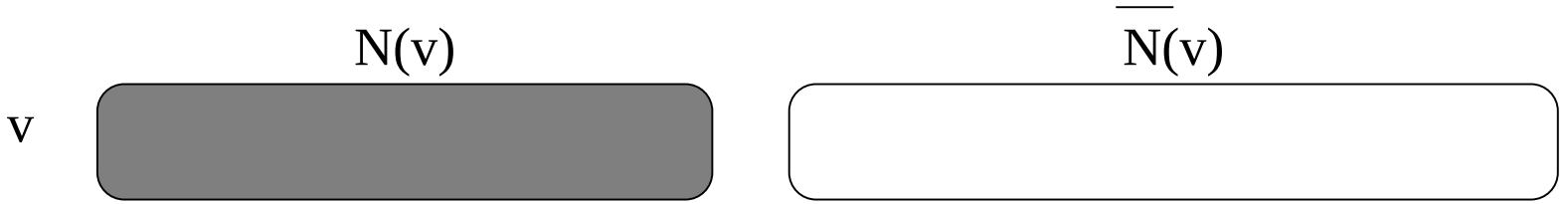
$\Rightarrow O(n + m \log n)$ time using vertex partitioning algorihtm.



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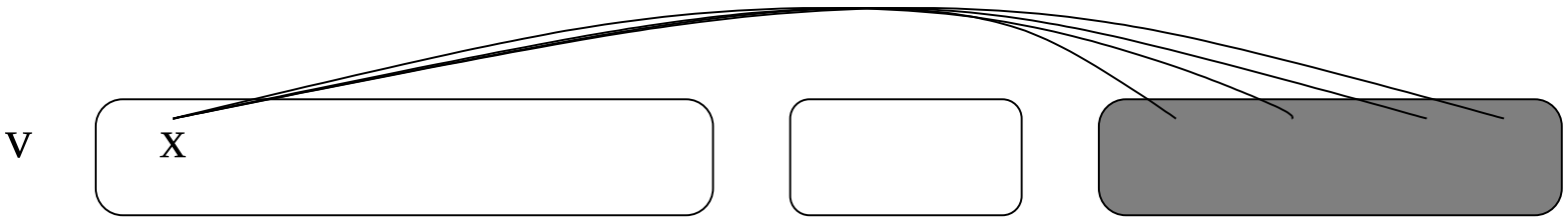

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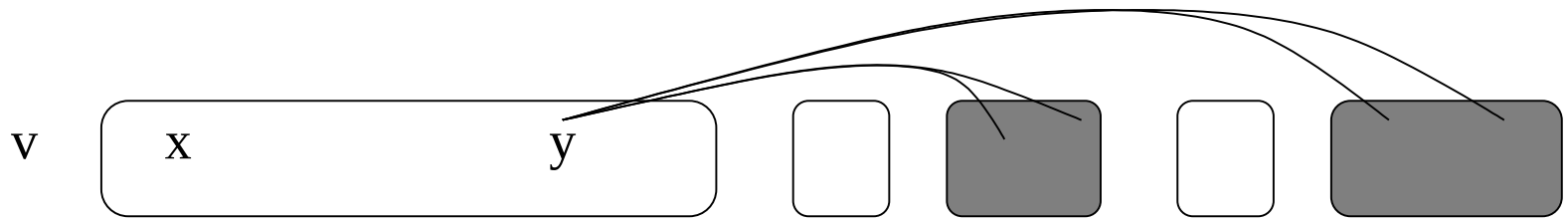

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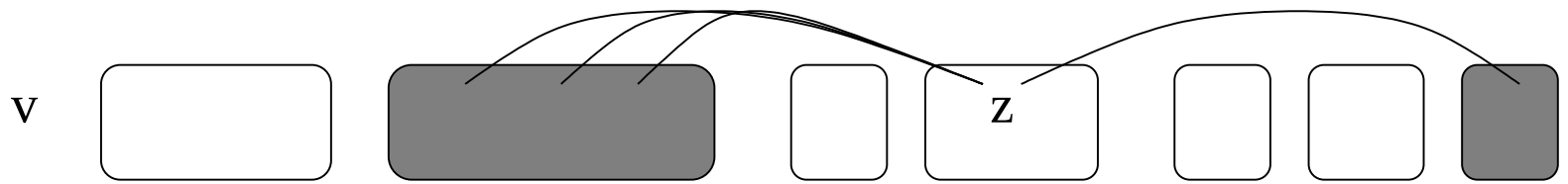
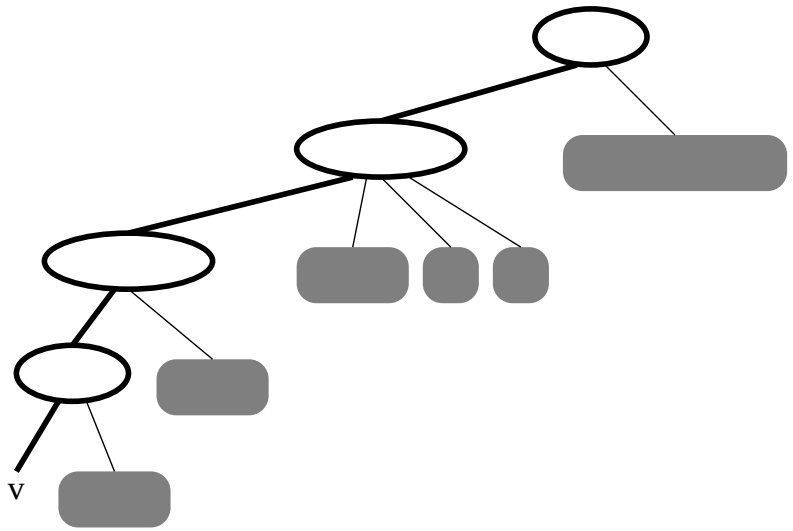


Computation of $MD(G_{/\mathcal{M}(G,v)})$

- The modules of $G_{/\mathcal{M}(G,v)}$ are linearly nested:
  any non-trivial module contains v



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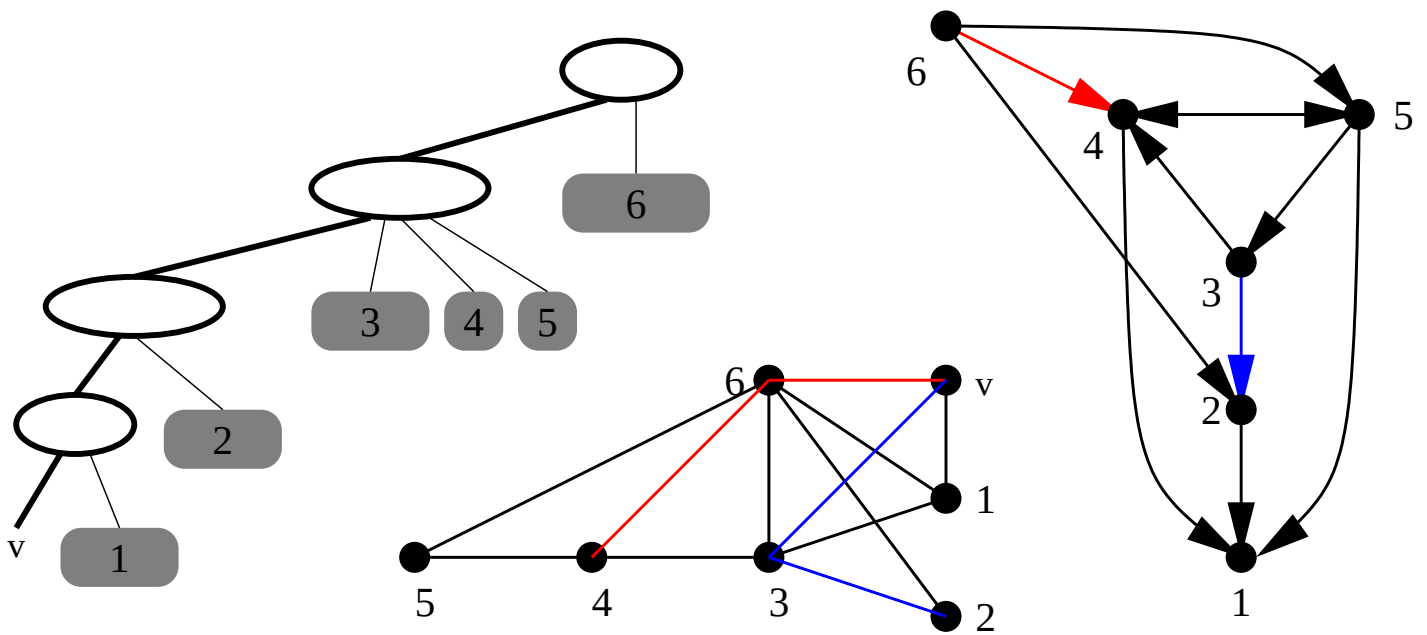
- The *forcing graph* $\mathcal{F}(G, v)$ has edge $\overrightarrow{xy}$ iff y separates x and v



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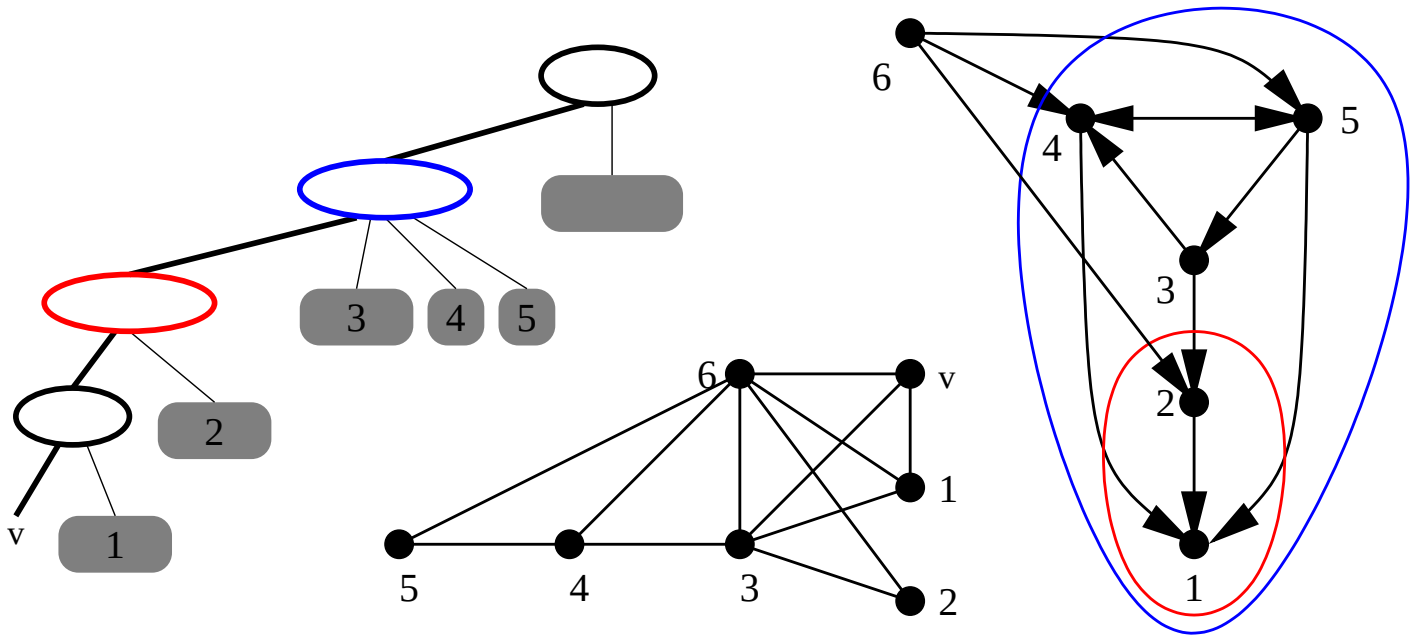

Complexity

- [Ehrenfeucht et al.'94] gives a $O(n^2)$ complexity.
- [MS00] gives a very simple $O(n + m \log n)$ algorithm based on vertex partitioning.
- [DGM'01] proposes a $O(n + m.\alpha(n, m))$ and a more complicated $O(n + m)$ implementation.

Other algorithms

- [CH94] and [MS94] present the first linear algorithms.
- [MS99] present a new linear time algorithm which extend to transitive orientation.

Still open

Find a simple linear time modular decomposition algorithm.

[Spinrad'03]

The new [linear time] algorithm [MS99] is currently too complex to describe easily [...] The first $O(n^2)$ partitioning algorithms were similarly complex; I hope and believe that in a number of years the linear algorithm can be simplified as well.

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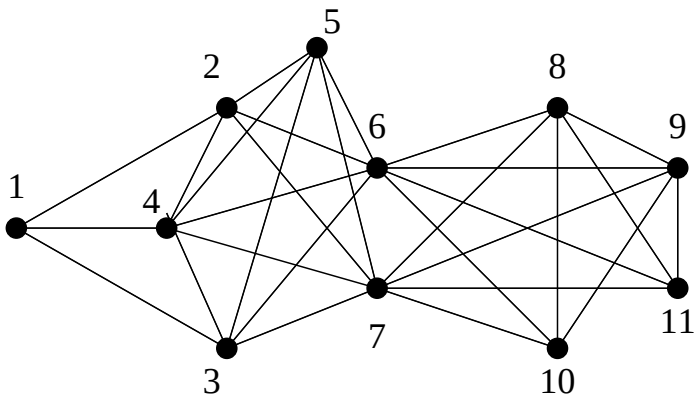
A **factoring pemutation** of a graph $G = (V, E)$ is a permutation of V in which any strong module of G is a factor. [Capelle 97]



1 2 3 4 5 6 7 8 9 10 11

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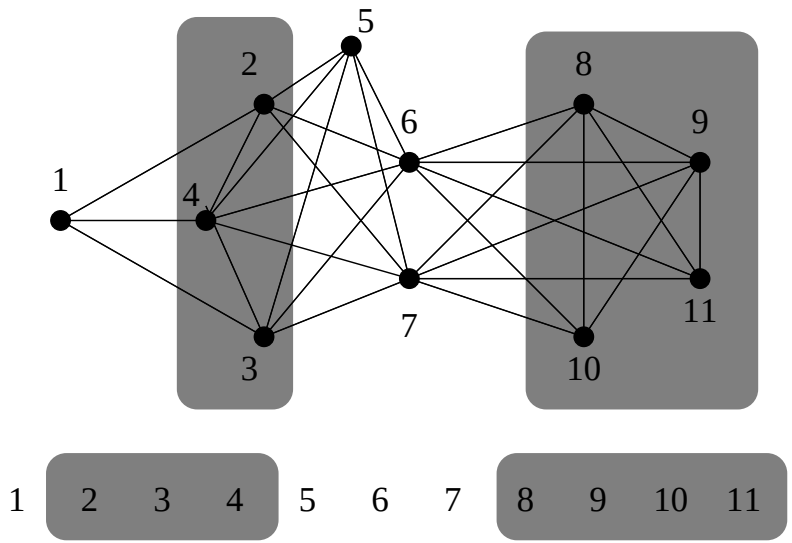

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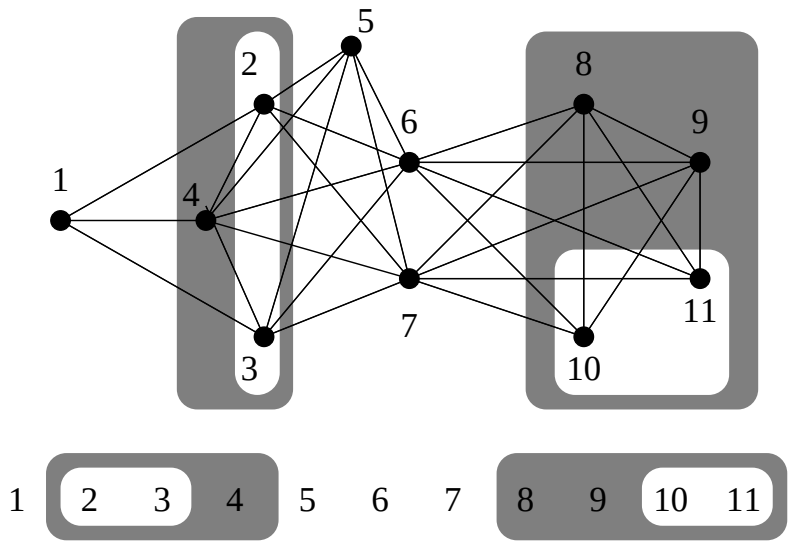

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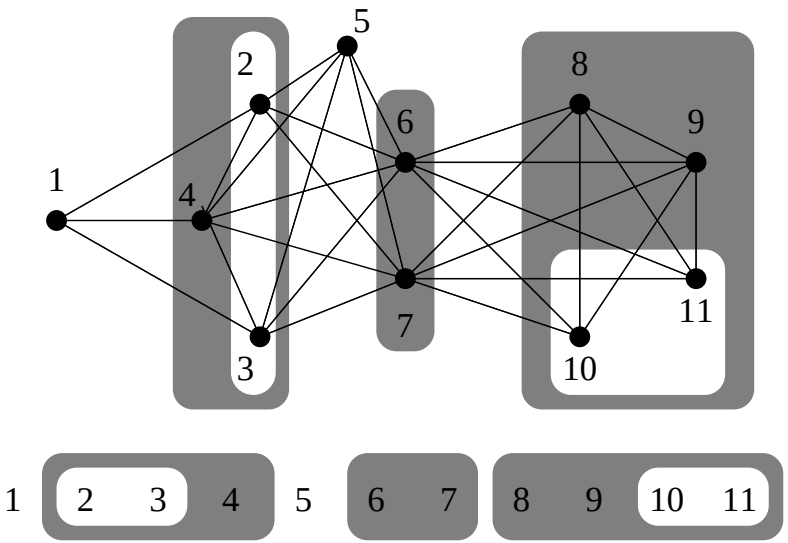

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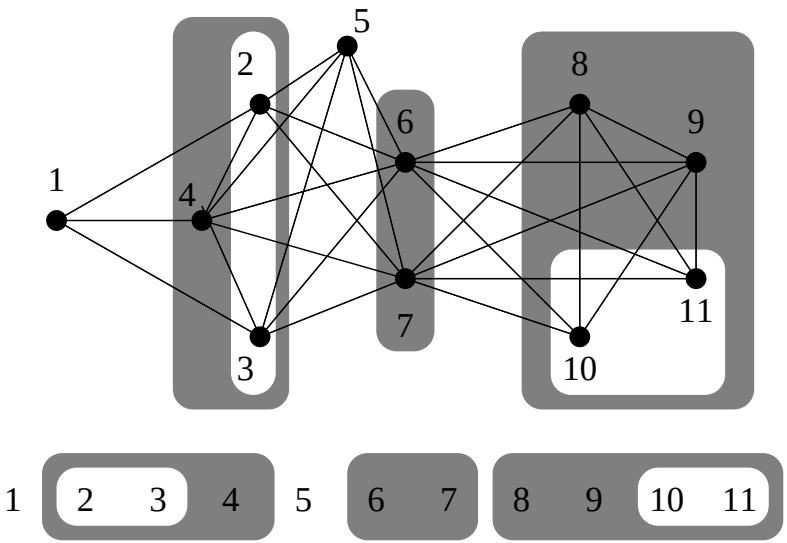

- From G to factoring permutation: $O(n + m \log n)$ [HPV99]

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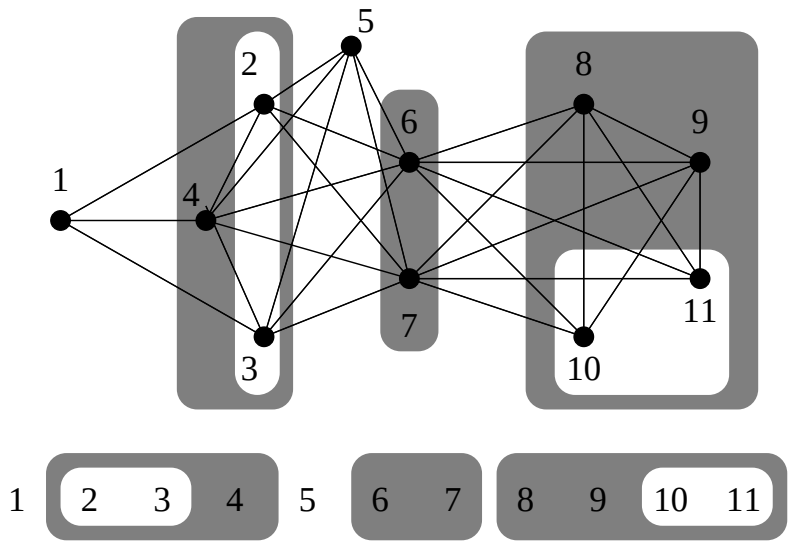
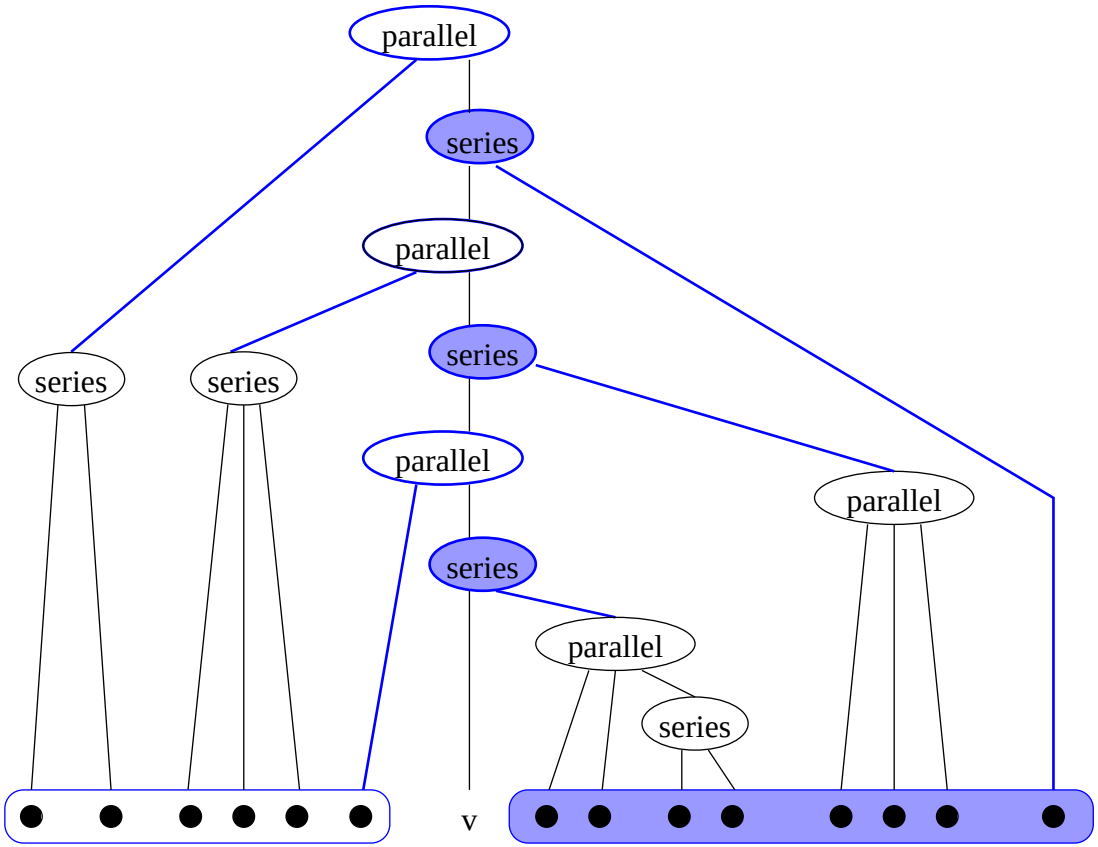


- From G to factoring permutation: $O(n + m \log n)$ [HPV99]
- From factoring permutation to $MD(G)$: $O(n + m)$ [CdMH01] [UY00] [BXHP05]

Factoring permutation (of cographs) via vertex partitionning

At each step, the partition can be refined into a factoring permutation.



Invariant

Any strong module is a factor of the partition.

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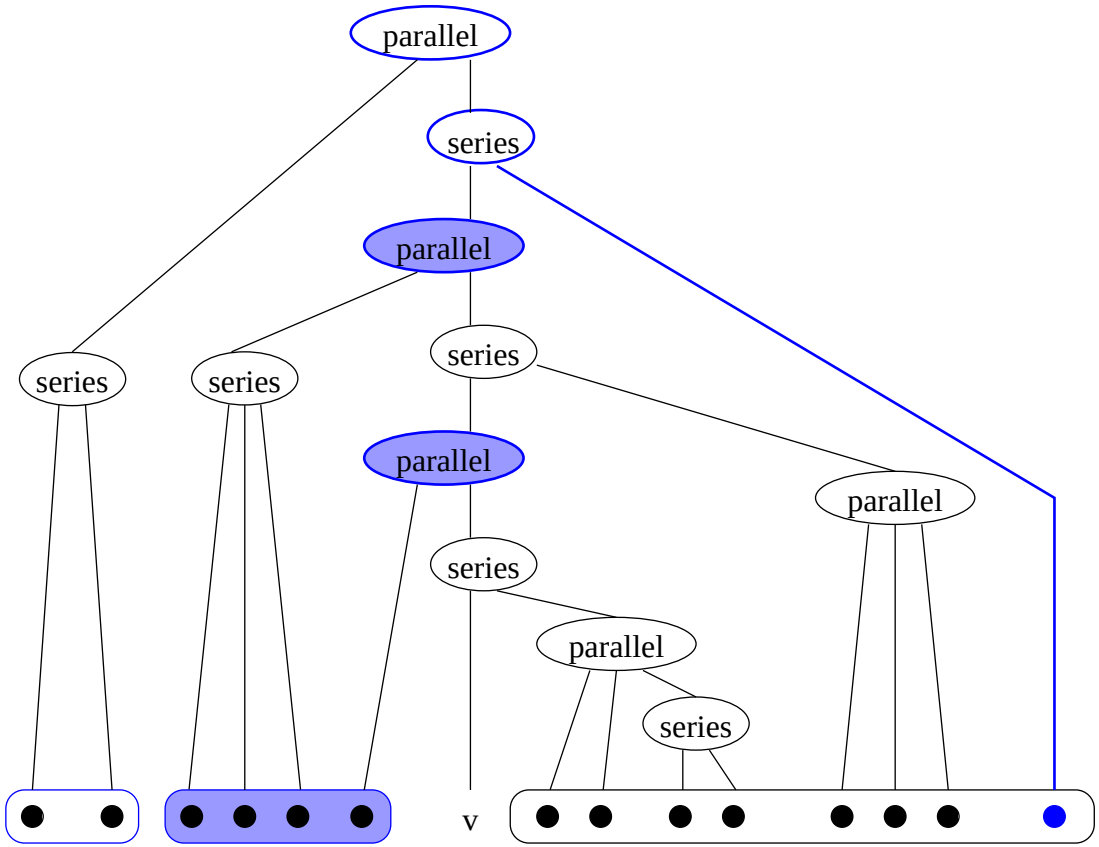

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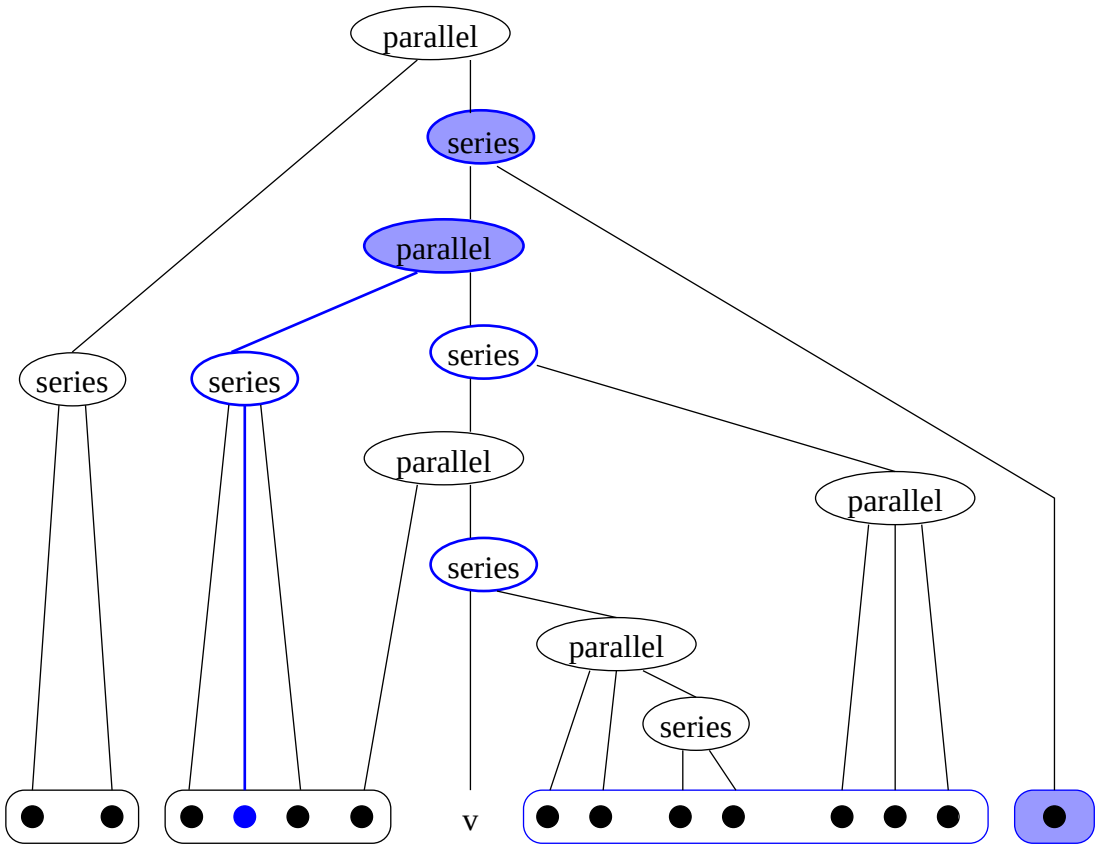

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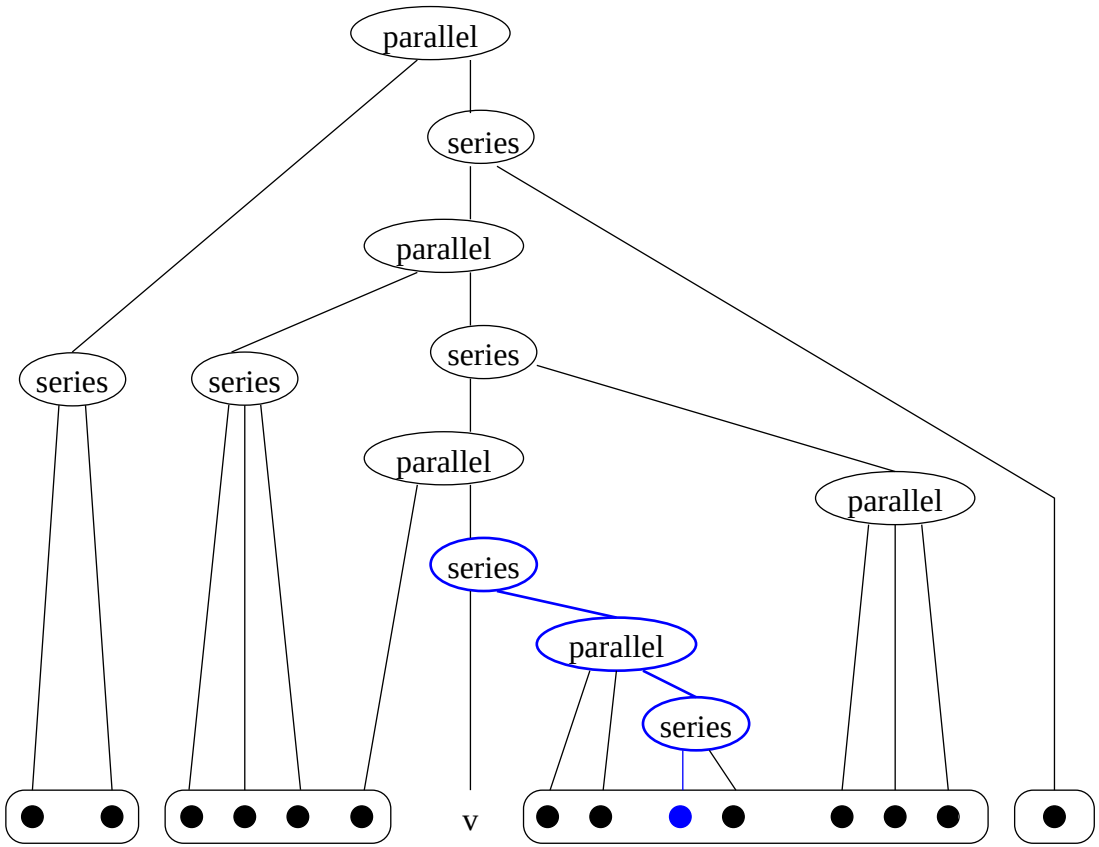

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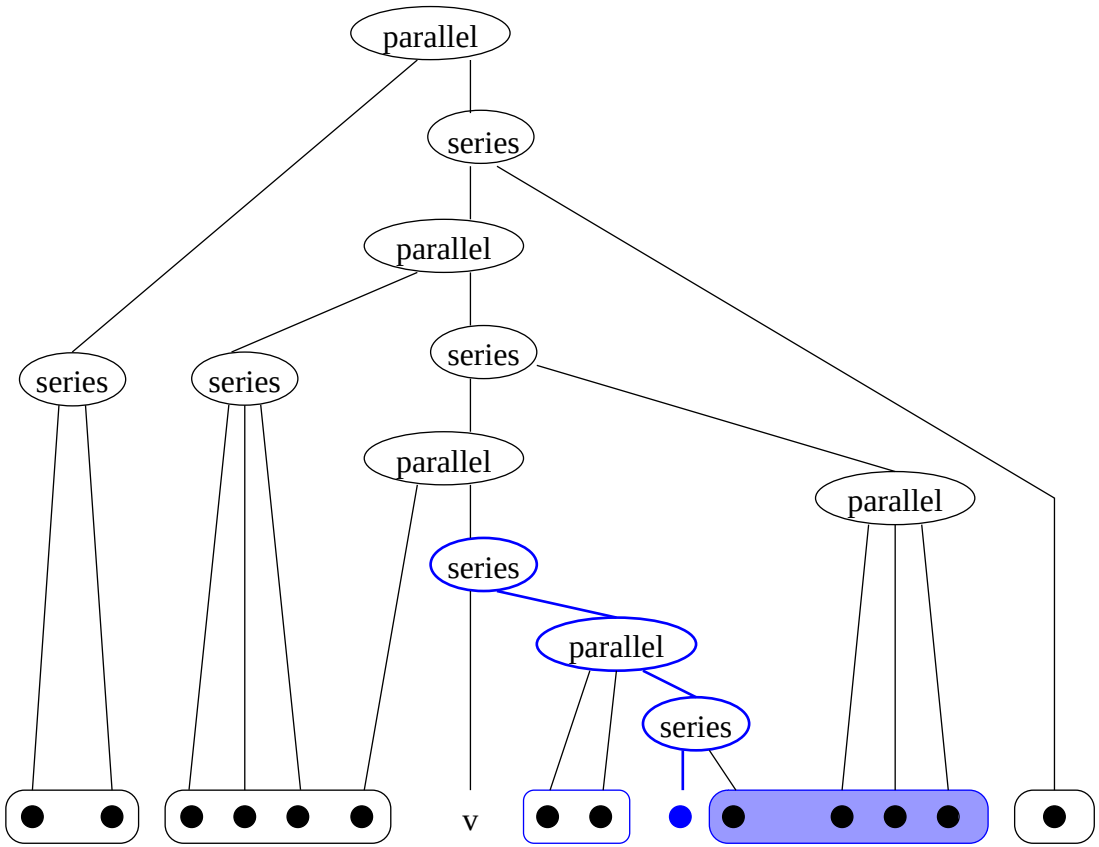

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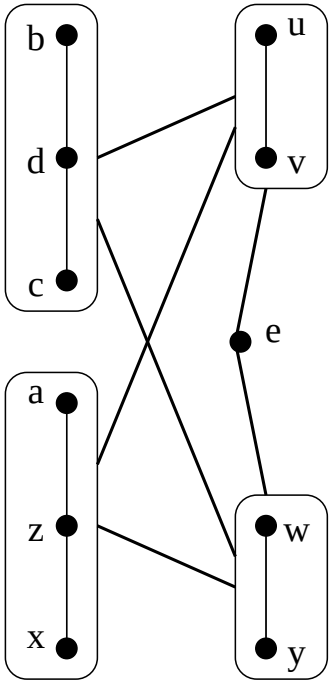
3. For some graph families, computing $MD(G)$ from a factoring permutation costs $O(n)$ time
4. Generalization for other decomposition like for example the split decomposition ?...

A cograph recognition algorithm [BCHP03]

1. $\sigma \leftarrow \text{LexBFS}(G)$
2. $\overline{\sigma} \leftarrow \text{LexBFS}^{-}(\overline{G}, \sigma)$
3. **If** σ and $\overline{\sigma}$ both have the NS-property **then**
   1. Answer "G is a cograph"
   2. Build $MD(G)$
4. **Else** Output a P_4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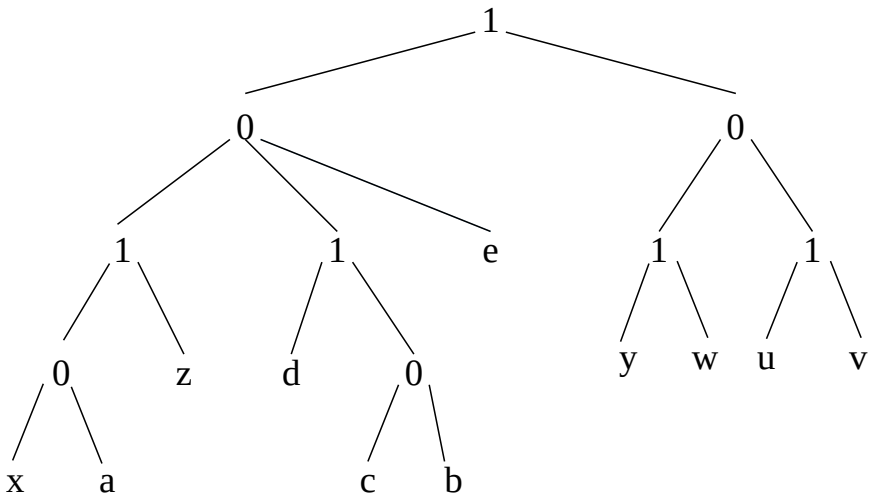
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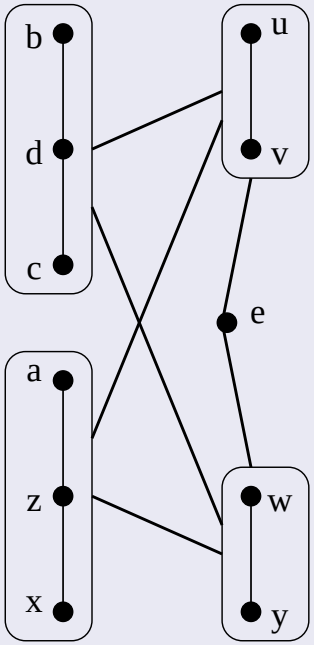
Computing a LexBFS ordering σ



x | y w z u v | a d c b e

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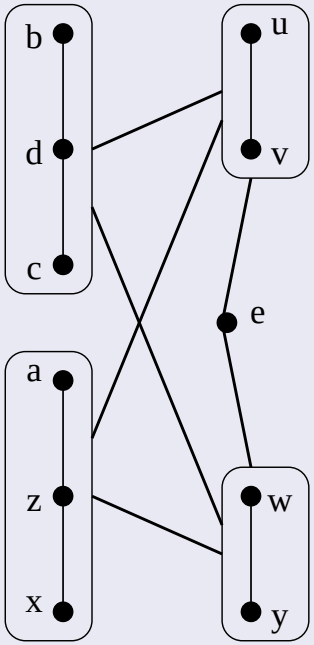

x | y w z u v | a d c b e

x y | w z | u v | a d c b e

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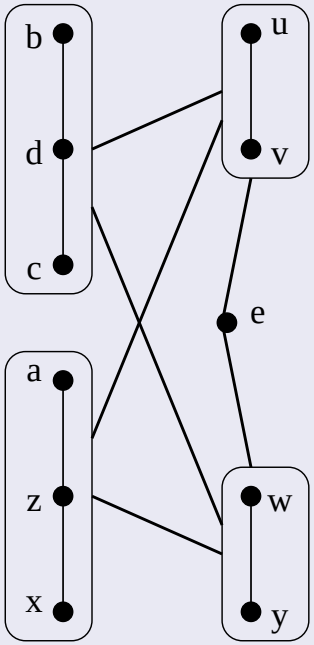

x | y w z u v | a d c b e

x | y | w z | u v | a d c b e

x | y | w | z | u v | a d c b e

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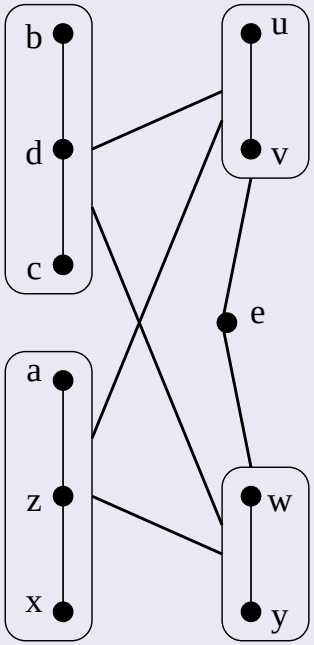

x	y	w	z	u	v	a	d	c	b	e
x	y	w	z	u	v	a	d	c	b	e
x	y	w	z	u	v	a	d	c	b	e
x	y	w	z	u	v	a	d	c	b	e

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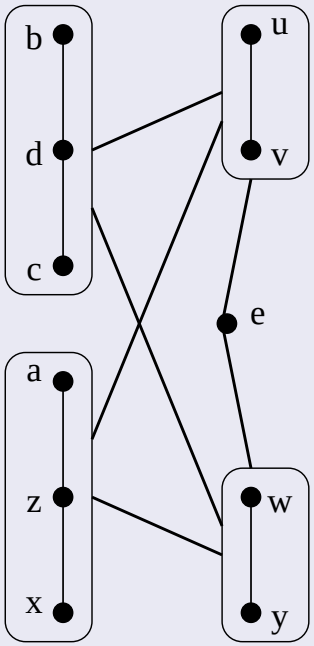
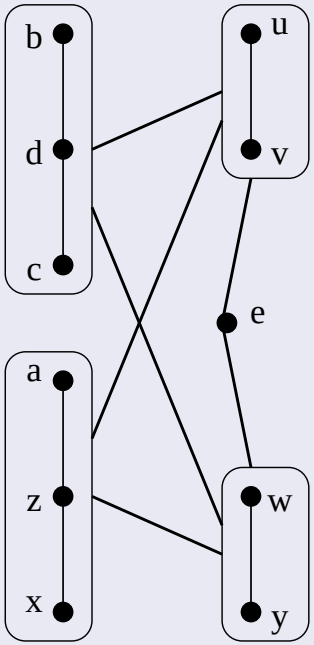


x	y	w	z	u	v	a	d	c	b	e
x	y	w	z	u	v	a	d	c	b	e
x	y	w	z	u	v	a	d	c	b	e
x	y	w	z	u	v	a	d	c	b	e
x	y	w	z	u	v	a	d	c	b	e

Computing $\overline{\sigma}$=LexBFS$^{-}(\overline{G}, \sigma)$



x | y w z u v | a | d c b e

x | a d c b e | y w z u v

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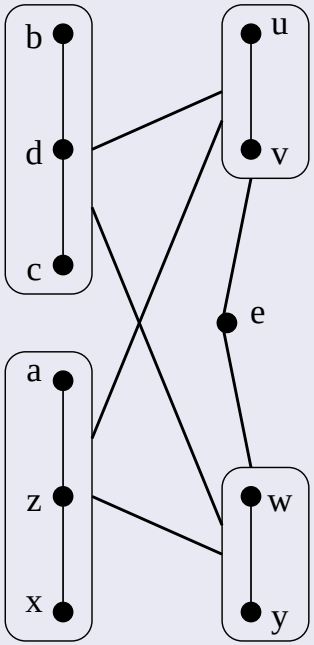

x | y w z u v | a d c b e

x | a d c b e | y w z u v

x a | d c b e | y w z u v

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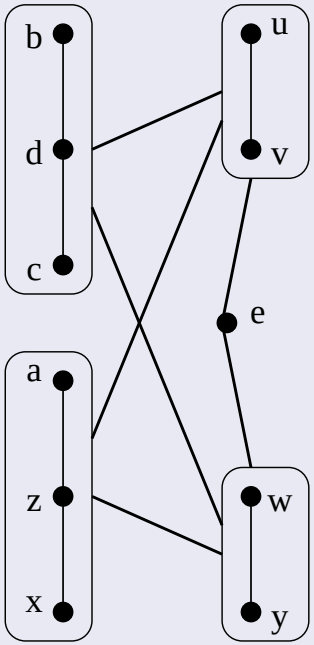
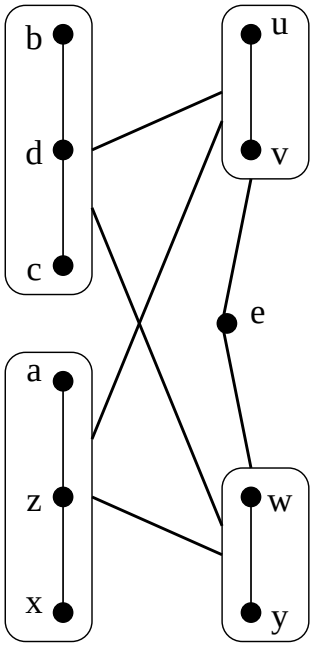
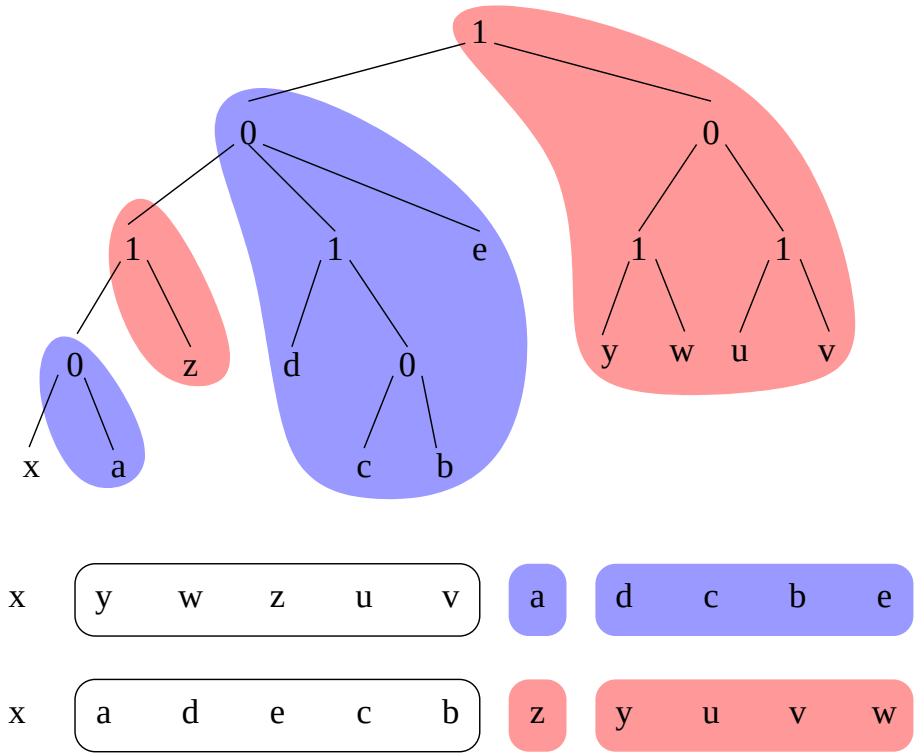


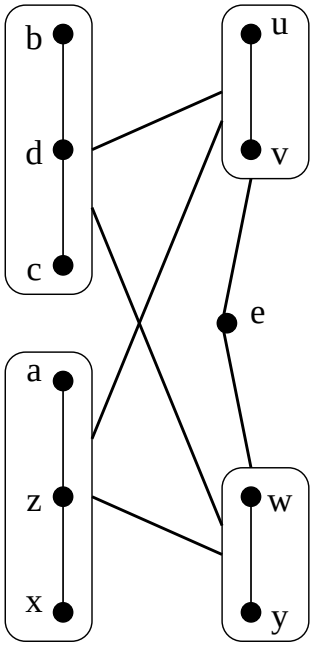
x	y	w	z	u	v	a	d	c	b	e
x	a	d	c	b	e	y	w	z	u	v
x	a	d	c	b	e	y	w	z	u	v
x	a	d	e	c	b	z	y	w	u	v

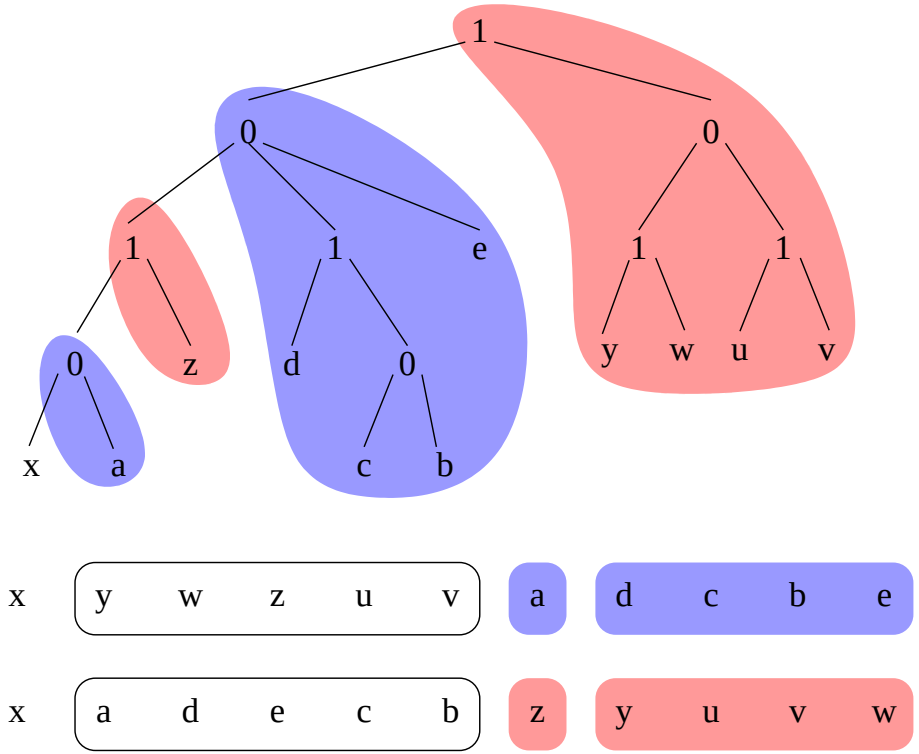




Lemma

$\mathcal{M}(G, x)$ is composed by the slices $S_i(x)$ of σ and $\overline{S_j}(x)$ of $\overline{\sigma}$.





Theorem

If G is a cograph, then $MD(G)$ can be retrieved from the slice structure of σ and $\overline{\sigma}$.

Generalisation to arbitrary graphs ?

1. There are many similarities between two-LexBFS-sweep algorithm and the linear implementation of Ehrenfeucht et al.'s algorithm [DGM01]
2. LexBFS is useful for the transitive orientation problem. Could it lead to a simple linear time algorithm for this problem ?

Miscellaneous : other works related to modular decomposition

1. Representation of dynamic graphs (cf. C. Crespelle's PhD-thesis)
   - cographs, permutation graphs ...

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1. Representation of dynamic graphs (cf. C. Crespelle's PhD-thesis)
   - cographs, permutation graphs ...
2. Perfect sorting by reversals
   - efficient parametrized algorithm based on the modular decomposition tree
   - characterization of the permutations having perfect parsimonious scenarios

Thank you...