Split vs Modular Decomposition
(the case of totally decomposable graphs)

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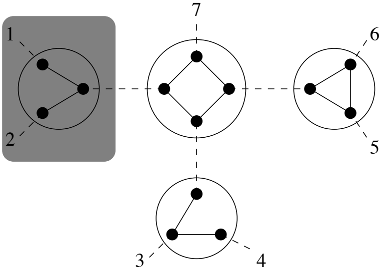
Joint work with E. Gioan (CNRS LIRMM)

Graph labeled tree

A *graph-labelled tree* is a pair $(T, \mathcal{F})$ with T a tree and $\mathcal{F}$ a set of graphs such that:

- each node v of degree k of T is labelled by a graph $G_v \in \mathcal{F}$ on k vertices
- there is a bijection ρ_v from the tree-edges incident to v to the vertices of G_v



Rooted graph labeled tree

A *rooted graph-labelled tree* is a pair $(T, \mathcal{F})$ with T a rooted tree and $\mathcal{F}$ a set of graphs such that:

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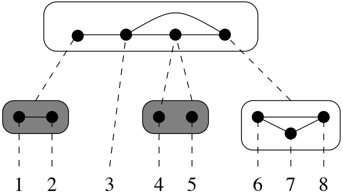
- there is a bijection ρ_v from the tree-edges between v and its children to the vertices of G_v



Given a rooted graph labelled tree $(T, \mathcal{F})$, the graph $G_M(T, \mathcal{F})$ has the leaves of T as vertices and

- $xy \in E(G_S(T, \mathcal{F}))$ iff $\rho_v(uv)\rho_v(vw) \in E(G_v)$, uv, vw tree-edges on the x, y-path in T and $v = lca_T(x, y)$.

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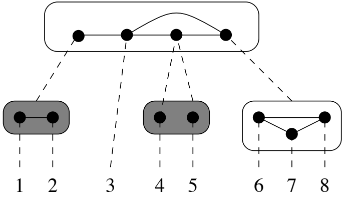

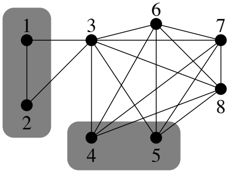

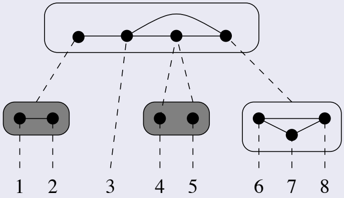

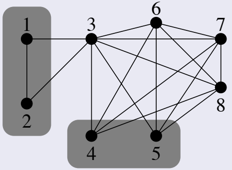

Modules

A subset of vertices M of a graph $G = (V, E)$ is a **module** iff

$$\forall x \in V \setminus M, \text{ either } M \subseteq N(x) \text{ or } M \cap N(x) = \emptyset$$

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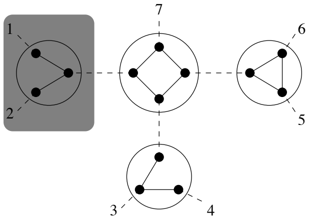

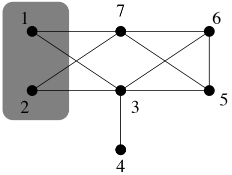

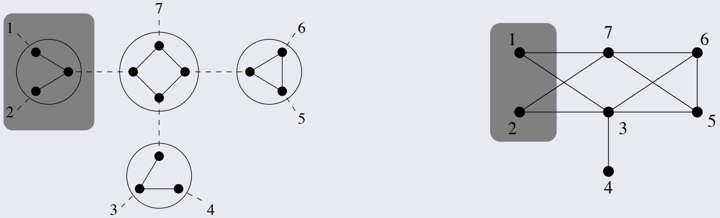

Split

A bipartition (A, B) of the vertices of a graph $G = (V, E)$ is a **split** iff

- $|A| \geqslant 2$, $|B| \geqslant 2$;
- for $x \in A$ and $y \in B$, $xy \in E$ iff $x \in N(B)$ and $y \in N(A)$.

Modular decomposition

Examples of modules

- any subset of vertices of the clique
- any subset of vertices of the stable

Split decomposition

Examples of splits

- any non-trivial bipartition of the clique
- any non-trivial bipartition of the $K_{1,n}$

Modular decomposition

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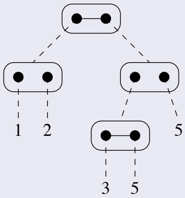
Totally decomposable graphs

- Cographs



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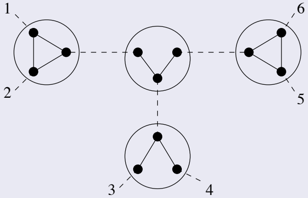
Totally decomposable graphs

- Distance hereditary grahs



Gallai'67 reformulated

For any graph G, there exists a unique rooted graph-labelled tree $(T, \mathcal{F})$ with a minumun number of nodes such that

1. $G = G_M(T, \mathcal{F})$ and
2. any graph of $\mathcal{F}$ is prime of degenerate for the modular decomposition.

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1. Definitions and preliminaries

2. Vertex modification of totally decomposable graphs

3. Conclusion and on-going work

Theorem (Corneil, Pearl and Stewart '85, Sharan and Shamir '04)

Let $G = (V, E)$ be a cograph. It can be tested in

- $O(|S|)$ **whether** $G + (x, S)$**, with** $x \notin E$ **and** $N(x) = S$**, is a cograph;**
- $O(|S|)$ *whether* $G - x$*, with* $S = N(x)$*, is a cograph;*
- $O(1)$ *whether* $G + e$*, with* $e \notin E$*, is a cograph;*
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Theorem (Tedder and Corneil '06, Gioan and Paul '07)

Let $G = (V, E)$ be a distance hereditary (DH) graph. It can be tested in

- $O(|S|)$ **whether** $G + (x, S)$**, with** $x \notin E$ **and** $N(x) = S$**, is a DH graph;**
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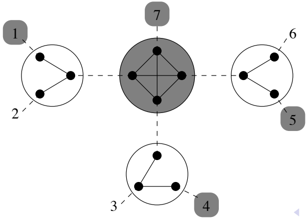
Let $(T, \mathcal{F})$ be a graph-labelled tree, and S be a subset of leaves of T. A node u of $T(S)$ is:

- **fully-marked** by S if any subtree of $T - u$ contains a leaf of S;



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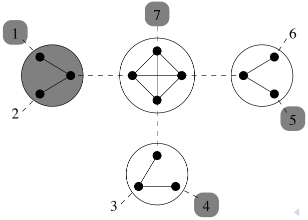
- **fully-marked** by S if any subtree of $T - u$ contains a leaf of S;
- **singly-marked** by S if it is a star-node and exactly two subtrees of $T - u$ contain a leaf $l \in S$ among which the subtree containing the neighbor v of u such that $\rho_u(uv)$ is the centre of G_u;



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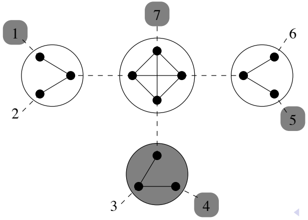
- **partially-marked** otherwise



Theorem (DH incremental characterization [Gioan, Paul '07])

Let G be a connected DH graph and $ST(G) = (T, \mathcal{F})$ be its split tree. Then $G + (x, S)$ is a DH graph iff:

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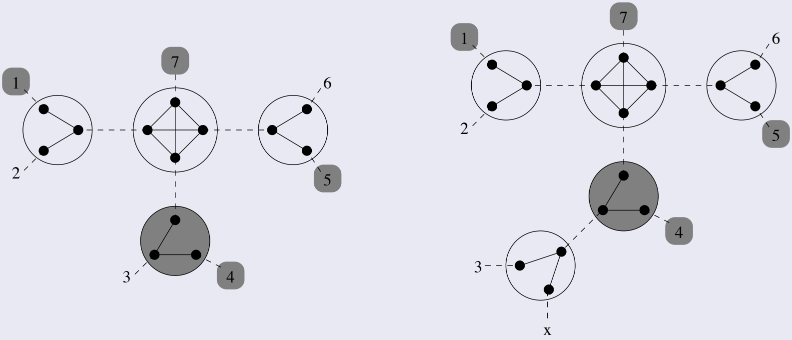
3. *If there exists a partially-marked node u, then any star node $v \neq u$ of $T(S)$ is oriented towards u if and only if it is fully-marked.*
4. *Otherwise, there exists a tree-edge e of $T(S)$ towards which any star node of $T(S)$ is oriented if and only if it is fully-marked.*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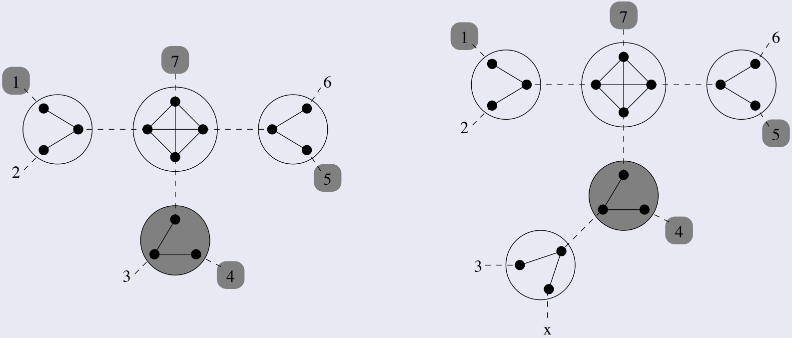
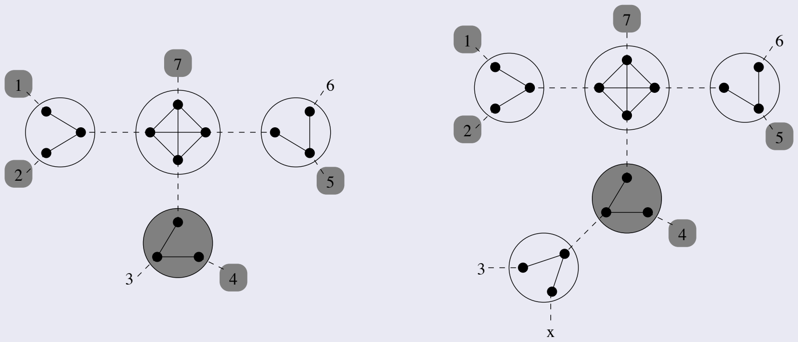


The insertion fails: the two singly-marked nodes are oriented towards the partially-marked node !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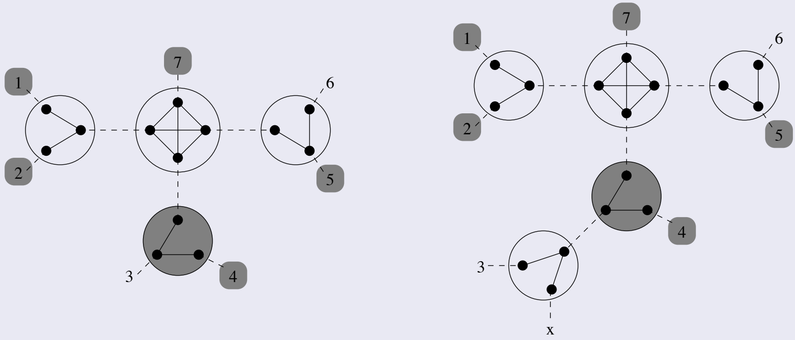

The insertion succeeds: in $G_S(T, \mathcal{F})$, we have $N(x) = S$

Insertion algorithm

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2. Check the mark-type of the nodes and look for an insertion node or edge;
3. Insert the node by either subdividing the insertion edge, or splitting the insertion node, or attaching x to the insertion node.

Cographs

Let $(T, \mathcal{F})$ be a rooted graph-labelled tree, and S be a subset of leaves of T. A node u of $T(S)$ is:

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Let G be a cograph and $MD(G) = (T, \mathcal{F})$ be its modular decomposition tree. Then $G + (x, S)$ is a cograph iff:

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4. *Otherwise, there exists a tree-edge $e = uw$ of $T(S)$ such that a parallel node $v \neq u$ of $T(S)$ is a descendant of u if and only if it is fully-accessible.*

Remarks

- There exists a characterization of cographs in terms of split tree.
- The graph-labelled tree representation of DH graphs yields an intersection model characterizing DH graphs.
- The edge-modification seems also to be possible using the split tree representation.

On-going work

- **Circle graphs and permutation graphs:** properties similar than for prime permutation graphs are observed for prime circle graphs.
- **Computation of the split tree:** An algorithm scheme like Ehrenfeucht et al's one for the modular decomposition can be design for the split decomposition.
- **Factorizing permutation:** such a concept would play the same role as for modular decomposition.
- **Generalization to other decompositions:** apply this graph-labelled tree approach to other decompositions.