Modular decomposition in kernelization

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Parameterized graph modification problems

- Can we update at most k vertices / edges of an input graph so get a property Π satisfied ?

Some examples

- FVS (or eq. treewidth 1 vertex deletion)
- odd cycle transversal
- minimum fill-in...

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Question:

Can modular decomposition help if we are interested in Π-modification problems where Π is related to some width-measure (clique-width, rank-width...) ?

Cluster editing as an appetizer

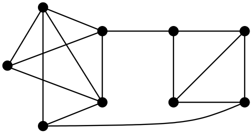
Modular decomposition in a nutshell

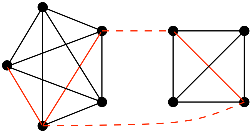
Cograph edge deletion kernel

Cluster Editing problem as an appetizer

Given a graph $G = (V, E)$ and an integer k, does there exist $F \subseteq V \times V$ such that

- $|F| \leqslant k$ and $G \triangle F$ is the disjoint union of cliques.





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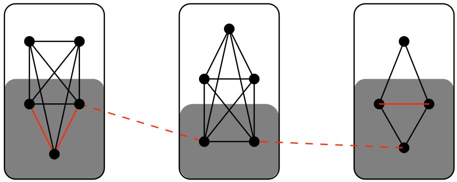
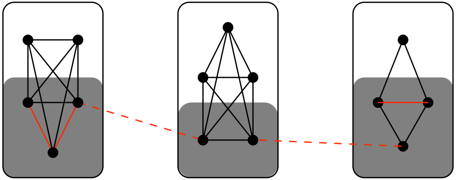


Structure of the solution

Observation 1: in each cluster, the non-affected vertices form a set of adjacent twins (or critical clique)

Cluster Editing problem as an appetizer



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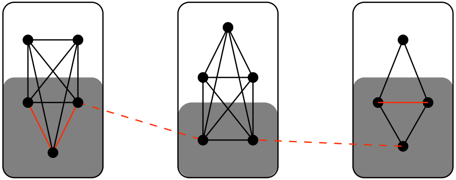

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Rule: shrink every critical of size $> k+1$ down to $k+1$

$\Rightarrow O(k^2)$ vertex kernel

More generally

Lemma Let $\mathcal{F}$ be an hereditary graph class closed under adjacent twin addition. Then there exists an optimal solution to the $\mathcal{F}$-edition problem that preserves the critical cliques.

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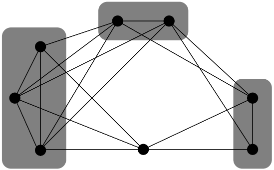
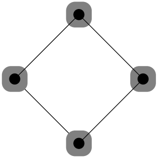
For cluster editing

$\longrightarrow$ edit all or none of the edges between two critical cliques





$\longrightarrow$ edit the edge set of the critical clique graph

$\Rightarrow 2k$ vertex kernel [Guo]

Cluster editing as an appetizer

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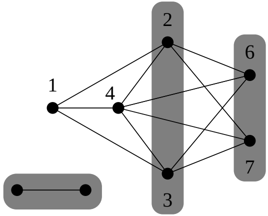
Cograph edge deletion kernel

A subset of vertices M of a graph $G = (V, E)$ is a module if

$$\forall x \in V \setminus M, \text{ either } M \subseteq N(x) \text{ or } M \cap N(x) = \emptyset$$



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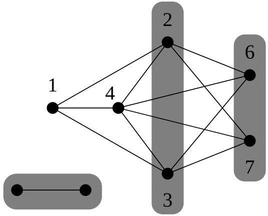

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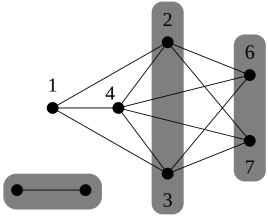

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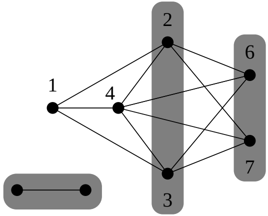

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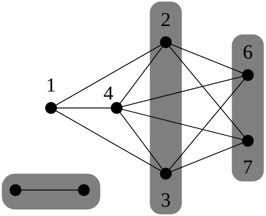

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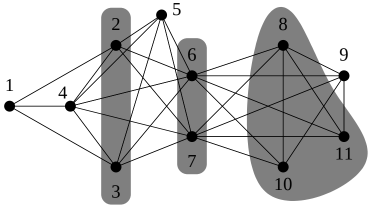
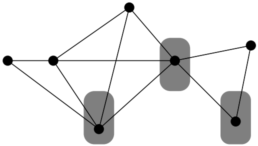
- any vertex subset of the complete graph (or the stable)

- A graph is prime if all its modules are trivial: e.g. the P_4.

- A graph is degenerate if any subset of vertices is a module: cliques and stables.

A partition $\mathcal{P} = \{M_1, \ldots M_r\}$ of the vertex set of a graph G is a modular partition of G if any part $M_i \in \mathcal{P}$ is a module of G.





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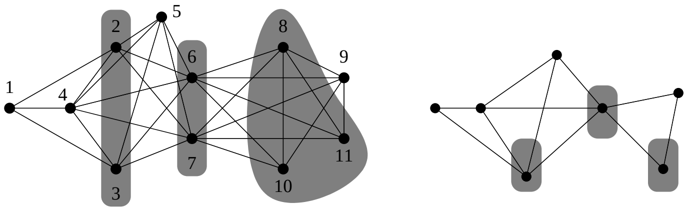

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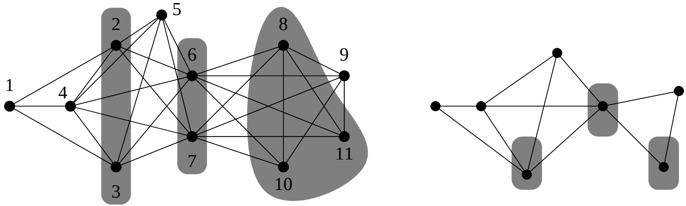

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Observation: The set of critical cliques form a modular partition and the corresponding quotient graph is the critical clique graph.

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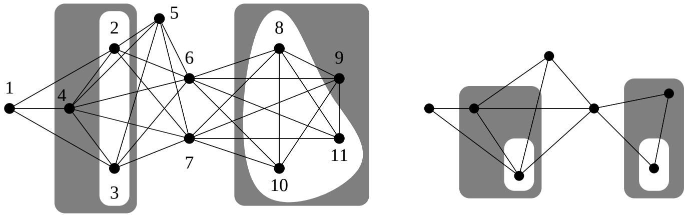

Lemma: $\mathcal{X} \subseteq \mathcal{P}$ is a module of $G_{/\mathcal{P}}$ iff $\cup_{M \in \mathcal{X}} M$ is a module of G.

Theorem [Gal'67,CHM81] Let G be a graph. Then either

1. G is not connected, or
2. $\overline{G}$ is not connected, or
3. $G_{/\mathcal{M}(G)}$ is a prime graph, with $\mathcal{M}(G)$ the modular partition containing the maximal strong modules of G.

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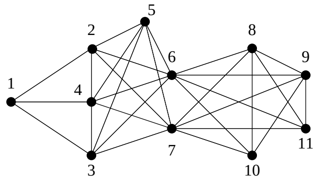

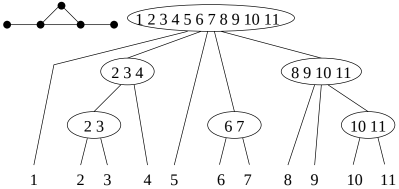

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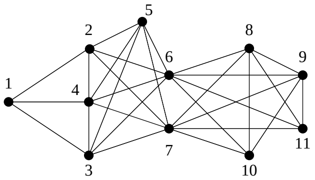
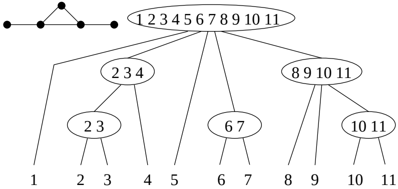




inclusion tree of strong modules = modular decomposition tree

MD-tree can be computed in $O(n+m)$

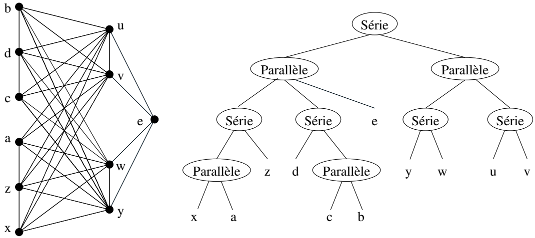
Some references

- Möhring, Radermacher. *Substitution decomposition for discrete structures and connections with combinatorial optimization*. Annals of Discrete Mathematics, 1984
- Habib, Paul. *A survey on algorithmic aspects of modular decomposition*. Computer Science Review, 2010
- and many others...

Cograph edge deletion kernel

G is a cograph if it can be built from the single vertex graph by:

- [disjoint union] $G = G_1 \oplus G_2$
- [series composition] $G = G_1 \otimes G_2$ (add all possible edges between G_1 and G_2)



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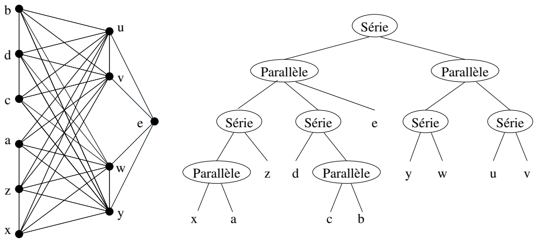

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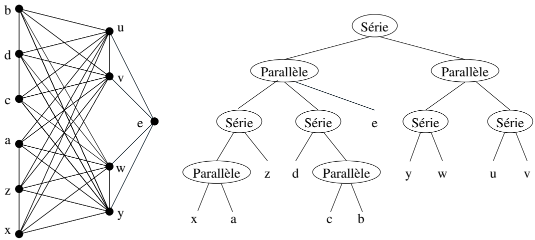

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- the P_4-free graphs
- the totally decomposable graphs by the modular decomposition

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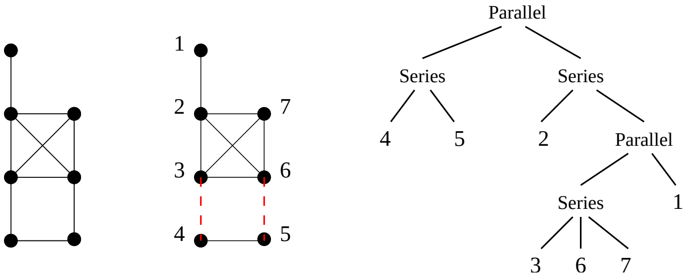

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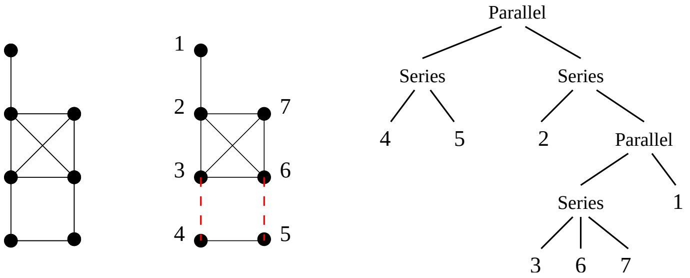
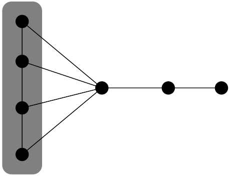


- use properties of modules
- use the (canonical) modular decomposition tree as witness structure of the solution

Observation: If M is a module and $abcd$ is a P_4, then either $\{a, b, c, d\} \subseteq M$ or $|\{a, b, c, d\} \cap M| = 1$



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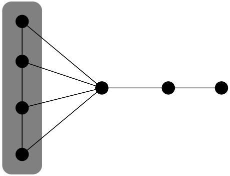

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- subgraphs induced by modules and quotient graphs

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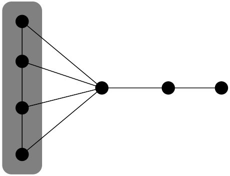

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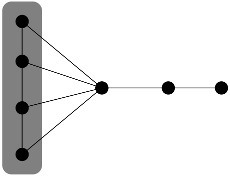

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$\longrightarrow$ same holds on tournaments for FAST

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Problem: improve the bound to quadratic

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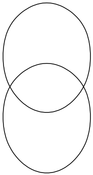
Modules already used in kernelization for:

- cluster editing, bicluster editing, min flip consensus tree, FAST, cograph edge modification...

What about generalizations ?

If M and M' are two overlapping modules then



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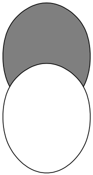

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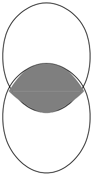

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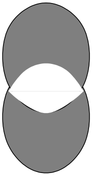

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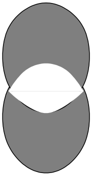

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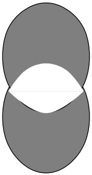

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Modular decomposition generalize to

- digraphs, permutations, k-structure...
- (weakly) partitive families, (weakly) bipartitive families (split decomposition)
- more generally tree-like set families

Example of problem related to these generalizations

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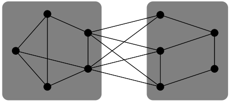
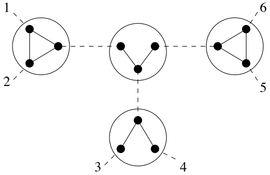
- G is a distance hereditary graph
- G is totally decomposable by the split decomposition



A split



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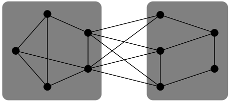
- G has rankwidth 1
  (BUT no finite set of forbidden subgraphs)



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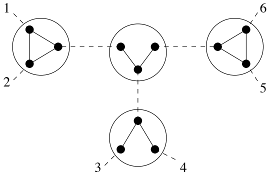

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