Important Sets and contraction-to-bipartite problem

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Joint work with
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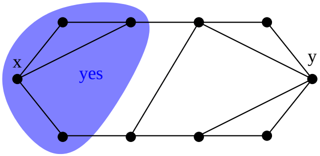
Ensembles et séparateurs Importants

Soit x, y deux sommets.
Un ensemble S est (x, y)-important si



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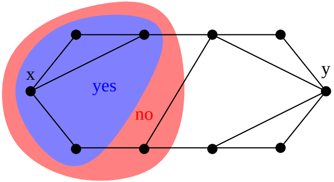

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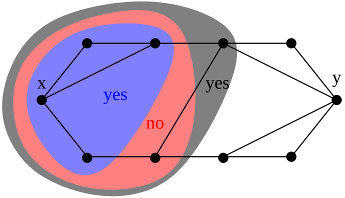

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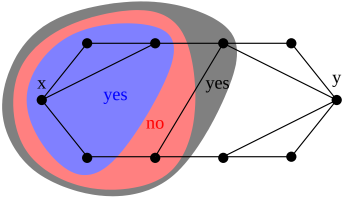

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$$\partial(S) = \{(u, v) \in E(G) \mid u \in S \vee v \notin S\}$$

Si S est un (x, y)-ensemble important, alors $\partial(S)$ est un (x, y)-séparateur (coupe) important

Remarque : pour des ensembles de sommets X et Y

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Observons que (1) ⇒ Théorème car

- on s'intéresse à $\mathcal{S}'$, le sous-ensemble des (x, y)-ensembles important avec $d_G(S) \leqslant k$
- donc $\sum_{S \in \mathcal{S}'} 4^{k-d_G(S)} \leqslant 4^k$
- et chaque terme de la somme est supérieur ou égal à 1

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$\Rightarrow$ $d_G(S' \cup S'') \leqslant \lambda$: contradiction avec la maximalité de S' et S''

Observation. tout (x, y)-ensemble important S vérifie $S^* \subseteq S$.

Soit $e = uv$ avec $u \in S^*$ et $v \notin S^*$,
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  - Par hyp. d'induction, nous avons $\sum_{S \in \mathcal{S}_{\bar{e}}} 4^{-d_G(S)} \leqslant 2^{-\lambda}/2$

Ensembles et séparateurs importants

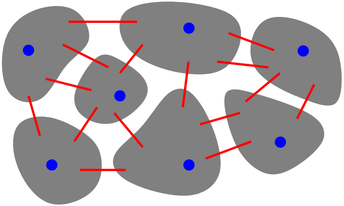
Multicoupe dans les graphes

Contraction-to-bipartite

- Iterative compression and 2-colorings
- Tree-width and well-connected set
- Important sets and irrelevant edges

Multicoupe paramétrée (multiway cut)

- Un graphe G et un ensemble T de sommets terminaux
- Un paramètre $k \in \mathbb{N}$
- Existe-t-il un ensemble S de k arêtes tel que chaque composante de $G \setminus S$ contient au plus un terminal de T ?



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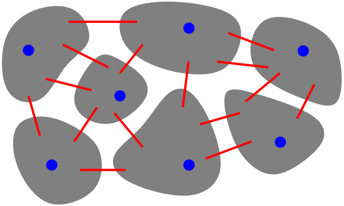

Théorème : Le problème Multicoupe paramétrée est FPT

Idée :

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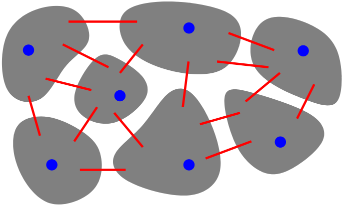

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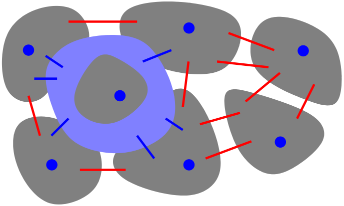

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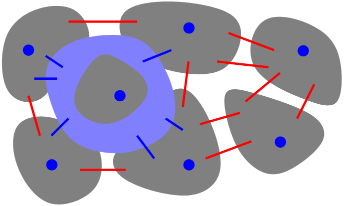
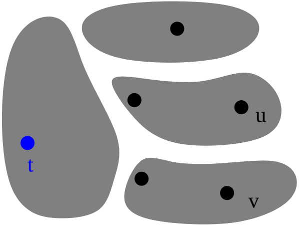


Lemme : $\forall t \in T$, MULTICOUPE PARAMÉTRÉE possède une solution contenant un $(t, T \setminus \{t\})$-séparateur important.



Soit X_t la composante connexe de $G \setminus S$ avec S une solution optimale

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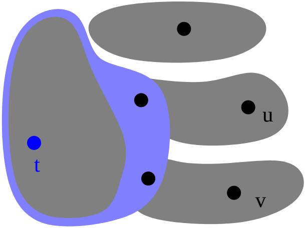

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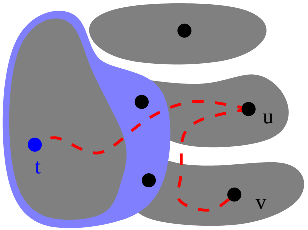

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- par définition, $|S'| \leqslant |S|$
- $\forall u \in T \setminus \{t\}$, $G \setminus S'$ ne contient pas de chemin entre t et u
- Si $G \setminus S'$ contient un chemin entre u et v, $u, v \in T \setminus \{t\}$, alors $G \setminus S$ contient ce chemin

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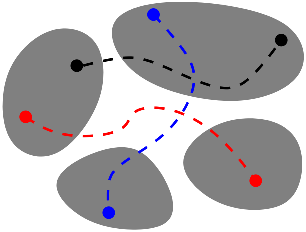
Théorème : Multicoupe paramétrée se résoud en $4^k.n^{O(1)}$

- le degré de branchement est borné par 4^k, le nombre maximum de $(t, T \setminus \{t\})$-séparateurs importants
- au plus k étapes de branchement

Multicoupe avec requêtes (Mutlicut)

- Un graphe G et un ensemble $\{(s_1, t_1), \ldots (s_l, t_l)\}$ de requêtes entre des paires de terminaux
- Existe-t-il un ensemble S de k arêtes tel que $G \setminus S$ ne contient aucun chemin entre s_i et t_i $(\forall i \in [l])$?



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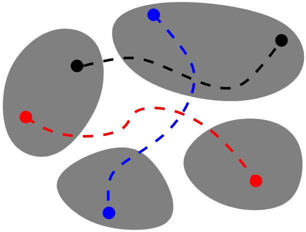

Théorème : Multicoupe avec requêtes est FPT paramétré par k et l

- Appliquer Multicoupe paramétrée sur toutes les partitions de $\{s_1, t_1, \ldots s_l, t_l\}$ (les terminaux sont les parties)

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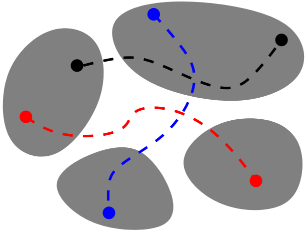

Théorème [Bousquet, Daligault, Thomassé] [Marx,Razgon]
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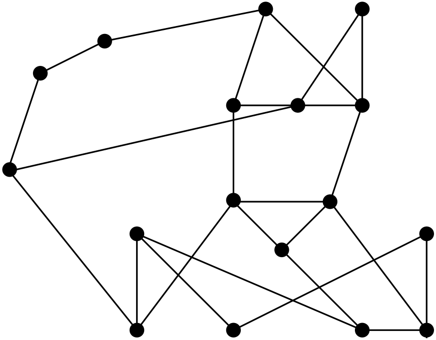
Contraction-to-Bipartite



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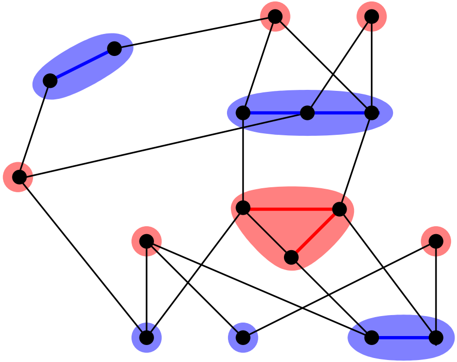

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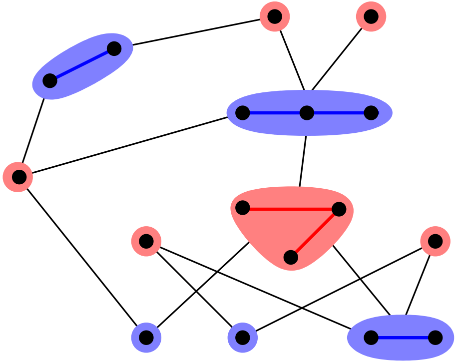
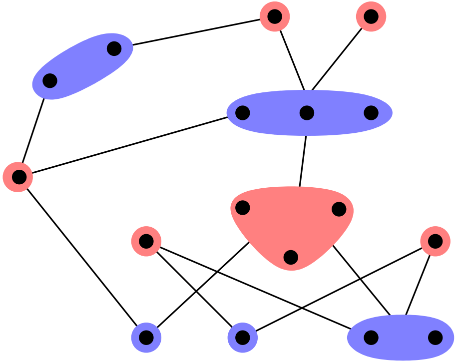


Given a graph G, find a subset $F \subseteq E(G)$ of at most k edges such that G/F is bipartite.

Contraction-to-Bipartite



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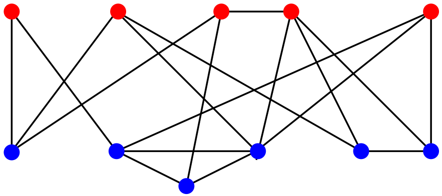

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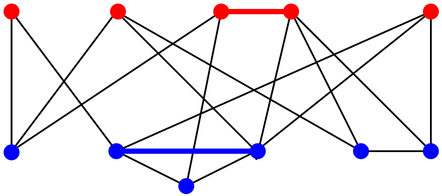

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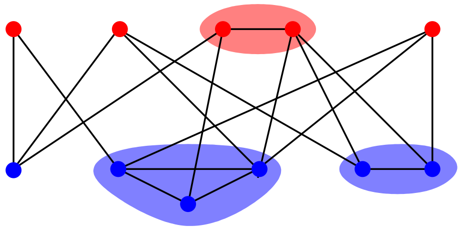

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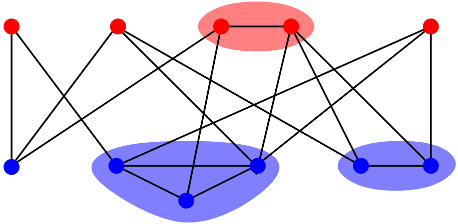

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- Let $\mathcal{M}_\Phi$ be the set of monochromatic components of the 2-coloring Φ, the cost of Φ is

$$c(\Phi) = \sum_{C \in \mathcal{M}_\Phi} (|C| - 1)$$

Lemma $G = (V, E)$ has a 2-coloring Φ of cost at most k iff there exists a set $F \subseteq E$ of at most k edges such that G/F is bipartite.

Skip Proof

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$\Leftarrow$ Observe that

$$V(G/F) \sim \{C' \mid C' \text{ is a connected component of } G' = (V, F)\}$$
$$x_{C'} \sim C'$$

If Φ' is a proper 2-coloring of G/F, then set

$$\boxed{\Phi(x) = \Phi'(x_{C'}) \Leftrightarrow x \in C'}$$

CHEAP COLORING

Given a graph G, an integer k,

- decide whether G has a 2-coloring Φ of cost at most k.

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CHEAPER COLORING

Given a graph G, an integer k and a 2-coloring Φ of cost $k + 1$,

- find a 2-coloring Φ' of cost at most k or conclude such a coloring does not exist.

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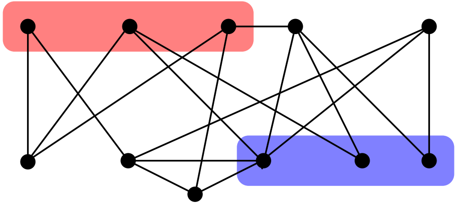

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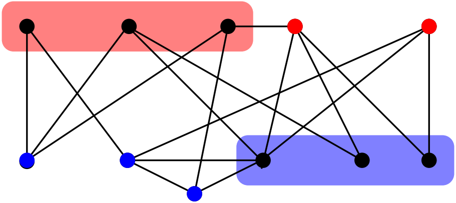

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Given a bipartite graph G, two integers k and t, and two subsets of vertices T_1 and T_2 such that $|T_1| + |T_2| \leqslant t$,

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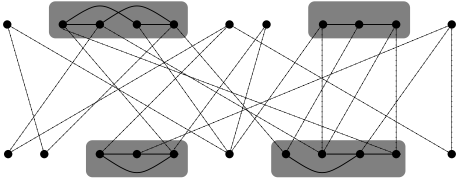

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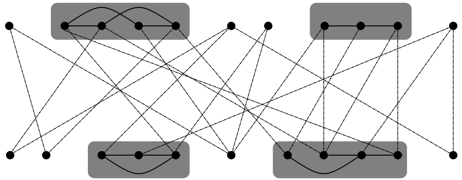
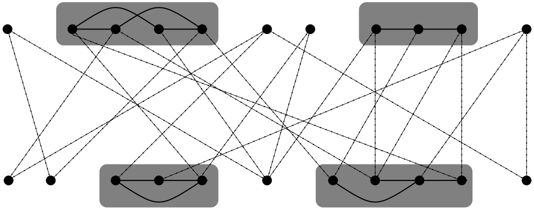


Lemma If there is an algorithm $\mathcal{A}$ for Cheap Coloring Extension that runs in time $f(k,t)n^c$, then is an algorithm for Cheaper Coloring that runs in time $4^{k+1}f(k, 2k+2)n^c$.

Proof   Skip Proof

Let T be the set of vertices spanned by the bad edges. For every bipartition (T_1, T_2) of X we construct a graph $G(T_1, T_2)$



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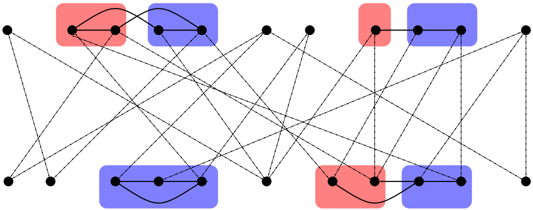

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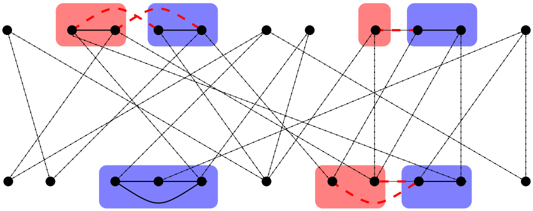

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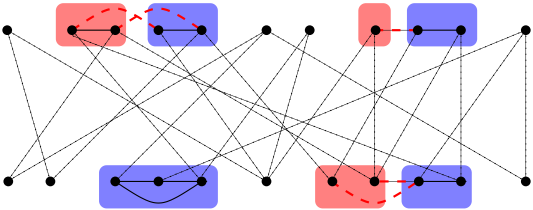

Claim. (G, k, Φ) is a <u>YES-instance</u> of CHEAPER COLORING iff $\exists$ a bipartition (T_1, T_2) of T such that $(G(T_1, T_2), k', t, T_1, T_2)$ is a <u>YES-instance</u> of CHEAP COLORING EXTENSION

Time complexity $\rightarrow 4^{k+1}.f(k, 2k+2).n^c$

CHEAP COLORING EXTENSION in FPT time

- if $tw(G)$ is small, then use Courcelle's Theorem

Lemma. There exists an algorithm that given an instance of CHEAP COLORING EXTENSION and a tree-decomposition of width ω, solves the instance in time $f(k, t, \omega).n$

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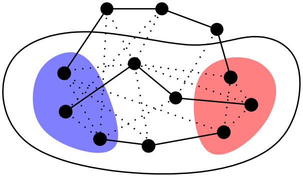
- Otherwise, we reduce to bounded tree-width
  - find an obstruction to small tree-width (→ a large well-connected set) in $c^{\omega} n^{O(1)}$ time
  - use the obstruction to identify an irrelevant edge (→ using important sets) in $f(k, t)n^{O(1)}$ time
  - reduce the graph by removing the irrelevant edge

well-connected sets as tree-width obstruction



A set X of vertices is p-connected if

- $|X| \geqslant p$ and
- $\forall\ X_1, X_2 \subseteq X$ such that $|X_1| = |X_2| \leqslant p$, there are $|X_1|$ vertex-disjoint paths in G between X_1 and X_2.

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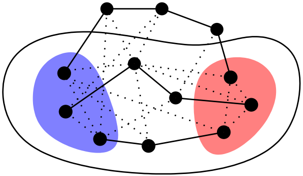

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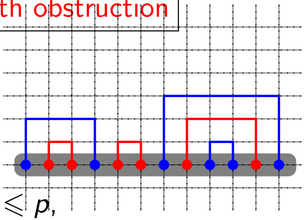
A set X is well-connected if it is $|X|/2$-connected.

Theorem [Diestel et al.'99]
If $tw(G) > \omega$, then G contains a well-connected set of size at least $2\omega/3$

Diestel et al. Proof

Skip Proof

p-connected sets as tree-width obstruction



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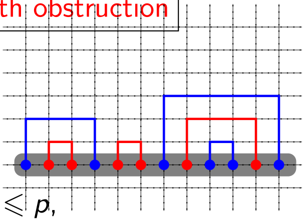

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Theorem [Diestel et al'99].
Let G be a graph and $\omega > 0$ be an integer

- If $tw(G) < \omega$, then G contains no $(\omega + 1)$-connected set of size at least 3ω. Proof of 1
- If G contains no externally $(\omega + 1)$-connected set of size at least 3ω, then $tw(G) \leqslant 4\omega$. Proof of 2

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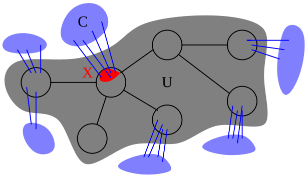

Let $U \subseteq V(G)$ be maximal such that:

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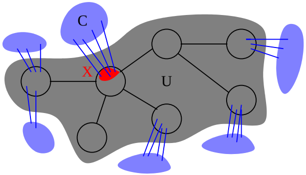

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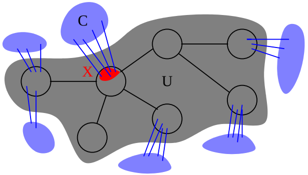

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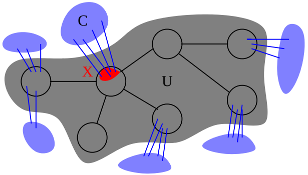

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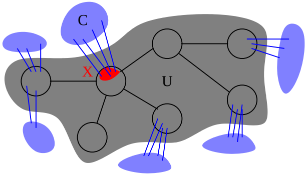

Claim. $U = V(G)$

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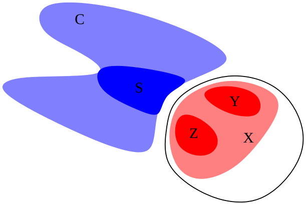

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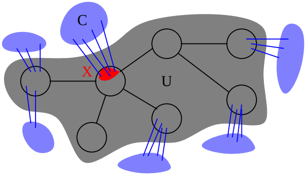

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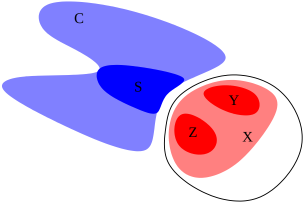

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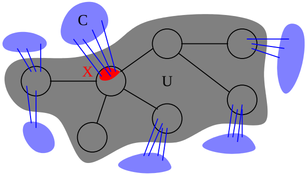

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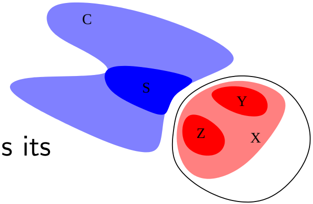
- Every component of $G - (U \cup S)$ has its neighbours in $X \cup S$

$\Rightarrow U$ is not maximal.



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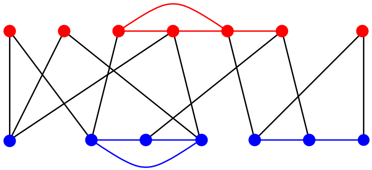
- We want to find an irrelevant edge e

$$(G, k, t, T_1, T_2) \in \textsc{Yes} \Leftrightarrow (G - e, k, t, T_1, T_2) \in \textsc{Yes}$$

Let Φ be a 2-coloring of G.

A good component of Φ is a connected component of the subgraph induced by the good edges.



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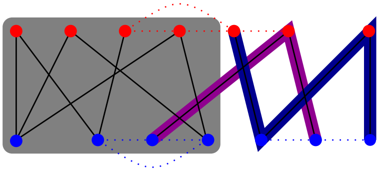

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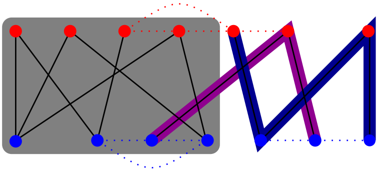

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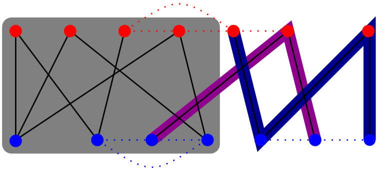

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Observation 2

If Φ has cost $\leqslant k$, G contains strictly less than $2k^2$ bad edges

(Since the k contracted edges cover at most $2k$ vertices)

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Observation 4

If C is good component of a cheapest (T_1, T_2)-extension Φ, then

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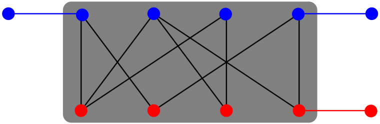

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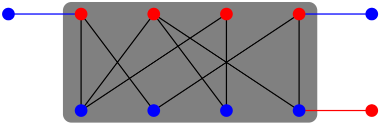


Flip the colors of a good component C st. $(T_1 \cup T_2) \cap C = \emptyset$

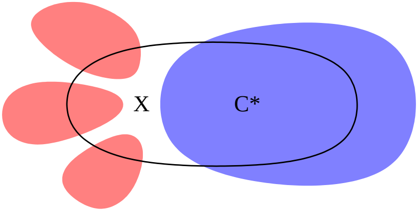
This is still a (T_1, T_2)-extension with smaller cost (by connectivity, at least one bad edge becomes good).

Remind X be our well-connected set of size $2(4k^2)t4^{4k^2}+2$

Lemma Let Φ be a cheapest (T_1, T_2)-extension of $G - uv$ of cost at most k. There exists exactly one good component C^* such that

$$\boxed{|X \setminus C^*| \leqslant 2k^2}$$

(every other good component C satisfies $|X \cap C| \leqslant 2k^2$)



Claim Every good component C satisfies

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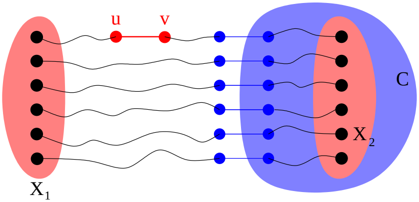

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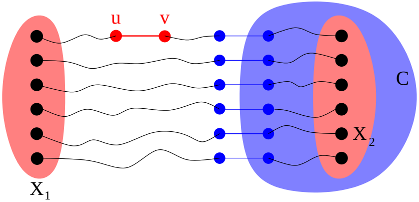

- Each of the X_1, X_2-paths contains a bad edge leaving C in G
- the removal of uv kills $\leqslant 1$ of these $2k^2 + 1$ paths
- so $G - uv$ has at least $2k^2$ bad edges: contradiction

For a cheapest (T_1, T_2)-extension Φ, we have:

- no bad edge in a good component
- strictly less that $2k^2$ bad edge
- an edge in a good component is irrelevant
- every good component intersect $T_1 \cup T_2$
- there is a good component C^* containing almost all the well-connected set X

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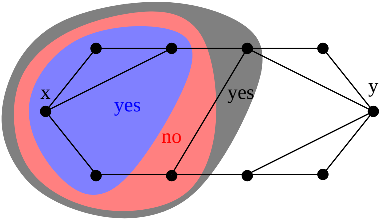
BUT

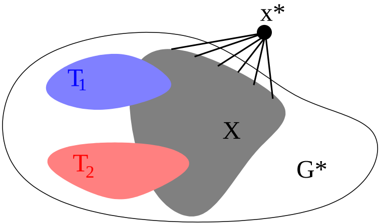
- How to identify $X \cap C^*$
- $X \cap C^*$ may be an independent set

Important Sets



Let x, y be two vertices.
A subset S is (x, y)-important if

- $x \in S$ and $y \notin S$
- $G[S]$ is connected
- $\nexists$ Y st.: $S \subset Y$, $d_G(Y) \leqslant d_G(S)$ and $G[Y]$ is connected

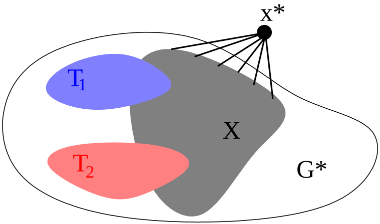


Let us consider $Z = \cup_{S \in \mathcal{S}} S$

with

$\mathcal{S} = \{S : \exists x \in T_1 \cup T_2, d_{G^*}(S) \leqslant 4k^2, S \text{ is } (x, x^*)\text{-important}\}$

($\rightarrow$ can be computed in time $4^{4k^2}.n^{O(1)}$)



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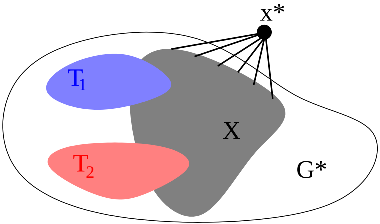
Lemma | Every edge uv of G st. $u \notin Z$ and $v \notin Z$ is irrelevant |



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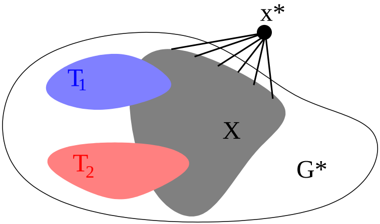
Let C^* be the big good component.

Claim | $u, v \in C^*$



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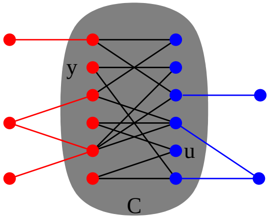
- if so, by Obs.3, uv is irrelevant

Claim $\boxed{u, v \in C^*}$

Assume $u \notin C^*$

- $\#$ bad edges $< 2k^2 \Rightarrow d_G(C) \leqslant 2k^2$
- $|C \cap X| \leqslant 2k^2 \Rightarrow d_{G^*}(C) \leqslant 4k^2$
- $\exists y \in C \cap (T_1 \cup T_2)$



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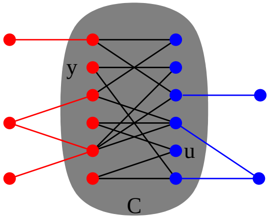

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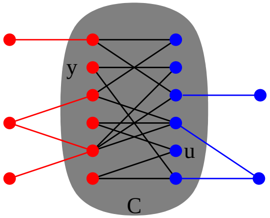

- $\exists$ an (y, x^*)-important set S st. $C \subseteq S$ with $d_{G^*}(S) \leqslant 4^{k^2}$
- then $C \subseteq S$ and $S \in \mathcal{S} \Rightarrow u \in Z$: contradiction.

Lemma G contains an edge uv st. $u \notin Z$ and $v \notin Z$

- Claim $\boxed{d_G(Z) \leqslant 4k^2.t.4^{4k^2}}$

  (Z is the union of at most $t.4^{4k^2}$ important sets, each of degree at most $4k^2$)

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- Claim $\boxed{Z \cap X \leqslant 4k^2.t.4^{4k^2}}$ (X the well-connected set)

  (each important set $S \in \mathcal{S}$ contains at most $4k^2$ vertices of X, since x^* is universal to X)

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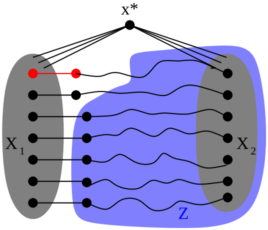
- Claim $\boxed{Z \cap X \leqslant 4k^2.t.4^{4k^2}}$ (X the well-connected set)
- Let (X_1, X_2) be a partition of X st. $|X_1| = |X_2|$, $Z \cap X \subseteq X_2$
  (this exists since $|X| \geqslant 2(4k^2).t.4^{4k^2} + 2$)



- every edge leaving X_1 contributes for one to $d_{G^*}(Z)$
- by pigeon-hole, one of these edges has its vertices out of Z

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Theorem

CONTRACTION-TO-BIPARTITE is fixed parameter tractable when parameterized by the number of edge contractions.

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OUFFFF !