Algorithmics of Modular decomposition

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Algorithms & Permutations Workshop
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Joint work with: A. Bergeron, S. Bérard, S. Bessy, B.M. Bui Xuan, C. Chauve, D. Corneil, F. Fomin, E. Gioan, M. Habib, A. Perez, S. Saurabh, S. Thomassé, M. Tedder, L. Viennot...

Modular decomposition of undirected graphs

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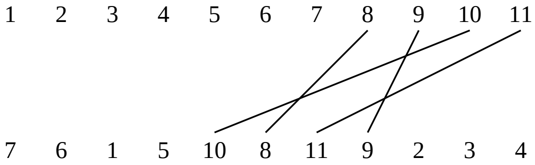

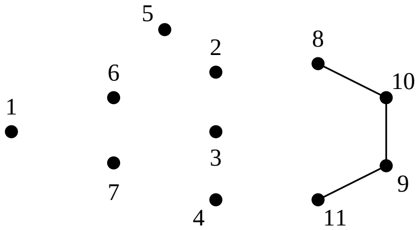

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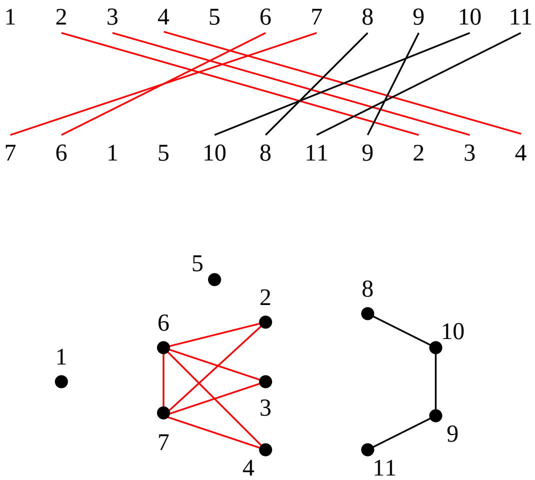

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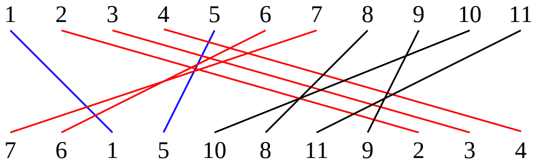

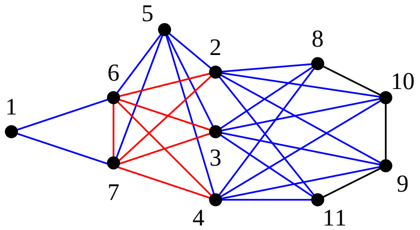

- Does a permutation graph have a unique representation ?

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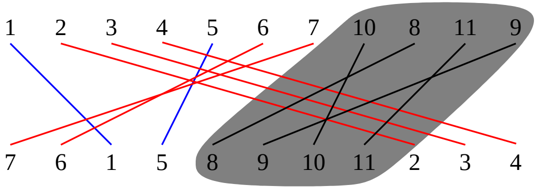
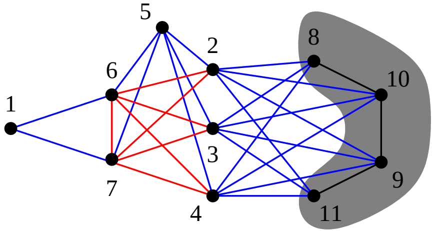
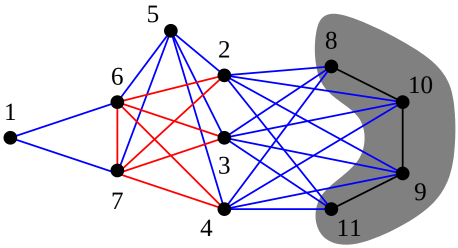




- Does a permutation graph have a unique representation ?

Modules

A subset of vertices M of a graph $G = (V, E)$ is a module iff
$$\forall x \in V \setminus M,\ \text{either } M \subseteq N(x) \text{ or } M \cap N(x) = \emptyset$$



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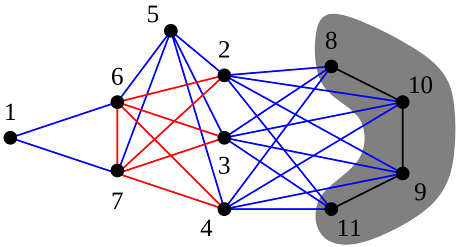

Examples of modules:

- connected components
- connected components of $\overline{G}$

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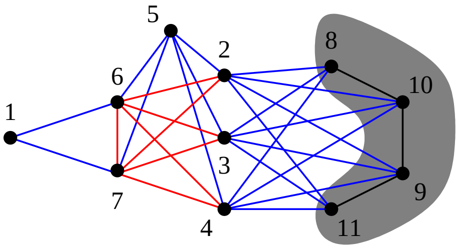
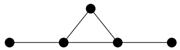


- A graph (a module) is prime is all its modules are trivial: e.g. the P_4.



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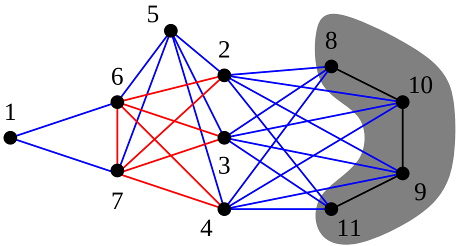

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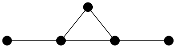

- A graph (a module) is degenerate if every subset of vertices is a module: cliques and stables.

Permutation graph recognition

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A permutation graph has a unique representation (up to reversal) iff it is prime.

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Recognition algorithm

- Recursively solve the problem on modules
- Solve the prime case (with linear time transitive orientation algorithm)

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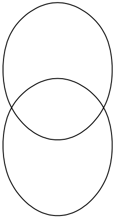
Theorem [McConnell and Spinrad'99]
The permutation graph recognition problem can be solved in $O(n + m)$ time

- we need a linear time modular decomposition algorithm

Partitive families

If M and M' are two overlapping modules then



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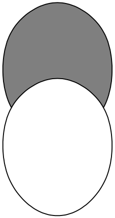

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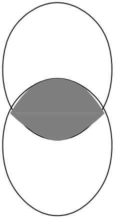

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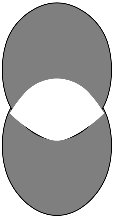

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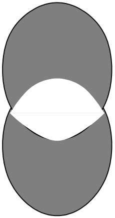

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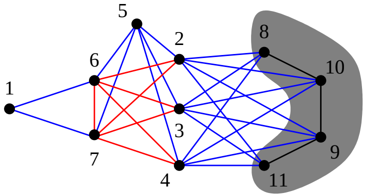
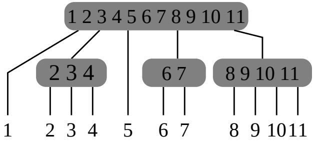
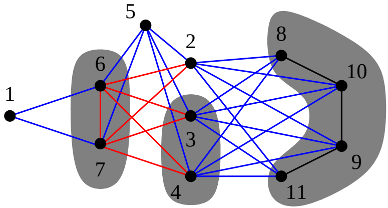




Strong modules are nested into an inclusion tree: the modular decomposition tree $MD(G)$

Modular partition and quotient graph

A partition $\mathcal{P}$ of the vertex set of a graph G is a modular partition if every part is a module of G.



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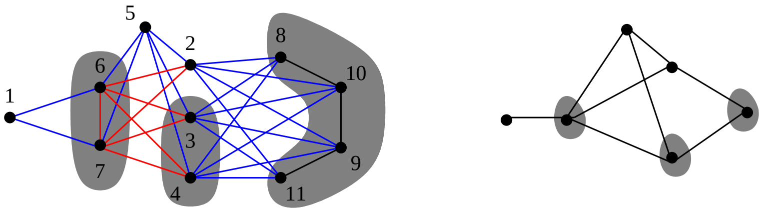

If $\mathcal{P}$ is a modular partition of G, the quotient graph $G_{/\mathcal{P}}$ is the induced subgraph obtained by choosing one vertex per part of $\mathcal{P}$.

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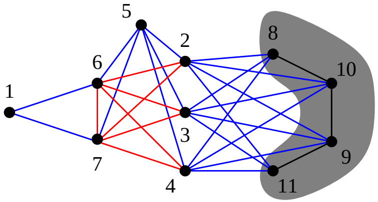

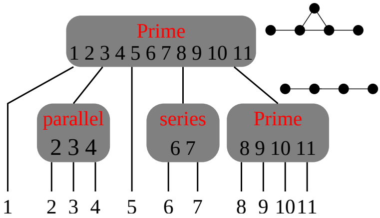

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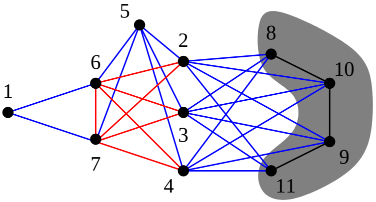

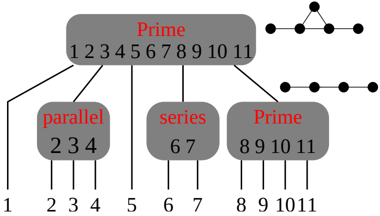

Observation: If a P_4 on $\{a, b, c, d\}$ overlap a module M, then

$$|M \cap \{a, b, c, d\}| = 1$$

Modular decomposition algorithms

- $O(n^4)$ [Cowan, James, Stanton'72]
- $O(n^3)$ [Blass, 1978], [Habib, Maurer'79]
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- $O(n + m\alpha(m, n))$ [Spinrad'92], [Cournier, Habib'93]
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- $O(n + m \log n)$ [Habib, Paul, Viennot'99] (factoring permutation), [McConnell, Spinrad'00]

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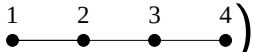
- $O(n + m)$ [Capelle, Habib'97] (factoring permutation) [Dahlhaus, Gustedt, McConnell'97], [Tedder, Corneil, Habib, Paul'08]

- other many others for variants of modular decomposition

Cographs - Totally decomposable graphs

Theorem: A graph is a cograph (a P_4-free graph )

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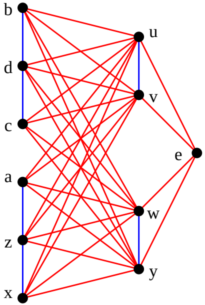

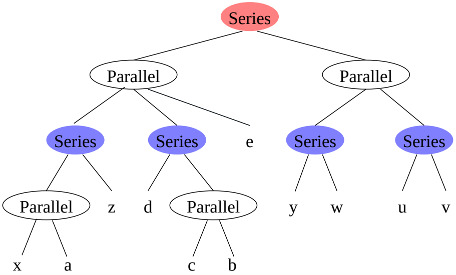

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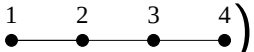
Theorem: A graph is a cograph (a P_4-free graph )

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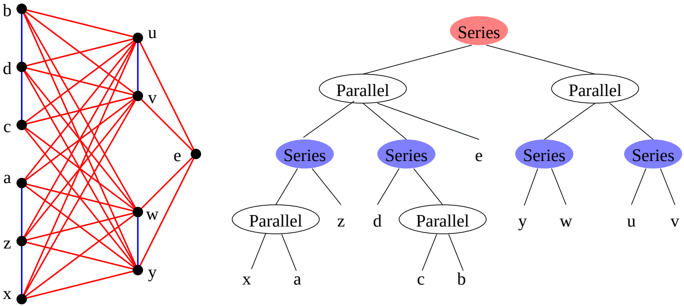

Cographs can be built from the single vertex with the disjoint union and series composition

Exercice: prove that cographs are permutation graphs

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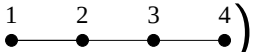
Theorem: A graph is a cograph (a P_4-free graph )
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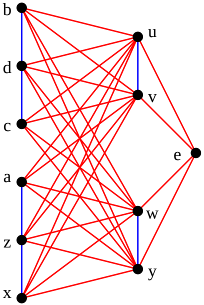

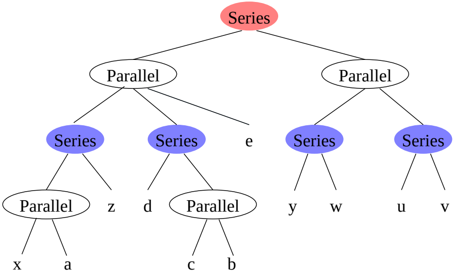

Linear time recognition algorithms

- incremental [Corneil, Pearl, Stewart'85]
- partition refinement [Habib, P.'05]
- LexBFS [Bretscher, Corneil, Habib, P.'08]

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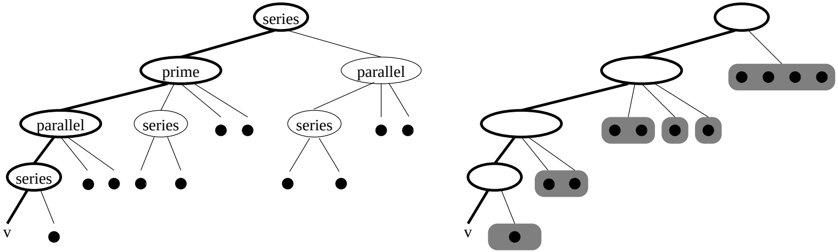
$\mathcal{M}(G, v)$ is the modular partition composed by

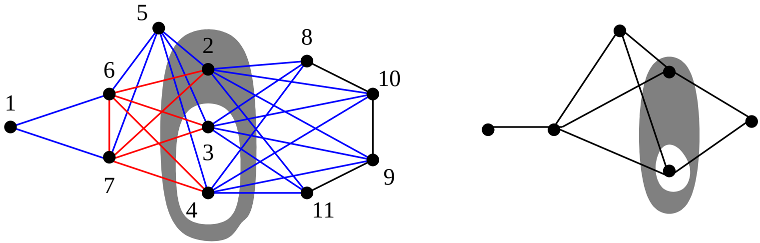
- $\{v\}$ and the maximal modules of G not containing v.



1. Compute $\mathcal{M}(G, v)$
2. Compute $MD(G_{/\mathcal{M}(G,v)})$
3. For each $\mathcal{X} \in \mathcal{M}(G, v)$ compute $MD(G[\mathcal{X}])$

Computation of $\mathcal{M}(G, v)$ (1)

Lemma [MR84] Let $\mathcal{P}$ be a modular partition of $G = (V, E)$.
$\mathcal{X} \subseteq \mathcal{P}$ is a module of $G_{/\mathcal{P}}$ iff $\cup_{M \in \mathcal{X}} M$ is a module of G.



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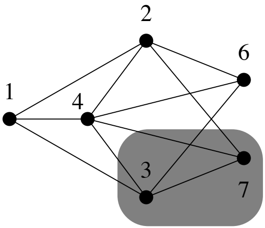
$$\exists y, z \in S \text{ with } xy \in S \text{ and } xz \notin E$$

We say that x separate y and z.



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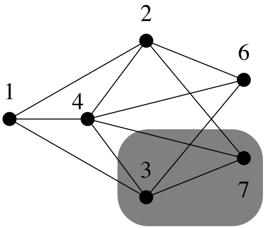

Lemma If x is a splitter for the set S, then any module M containing S must also contain x.

Computation of $\mathcal{M}(G, v)$ (2)

Lemma If v is a splitter of a set S, then for any module $M \subseteq S$ either $M \subseteq S \cap N(v)$ or $M \subseteq M \cap \overline{N}(v)$

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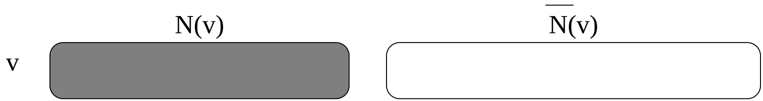
Lemma If v is a splitter of a set S, then for any module $M \subseteq S$ either $M \subseteq S \cap N(v)$ or $M \subseteq M \cap \overline{N}(v)$

- $O(n + m \log n)$ time using vertex partitioning algorihtm



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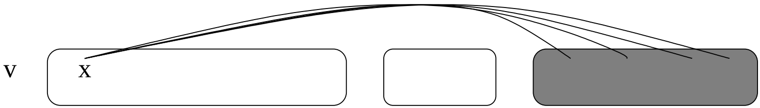

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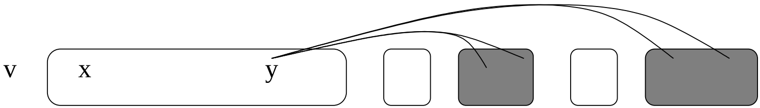

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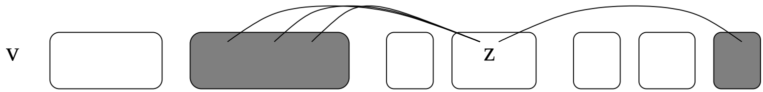

Computation of $MD(G_{/\mathcal{M}(G,v)})$ (3)

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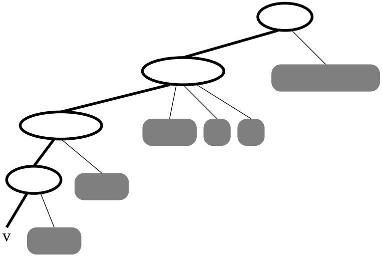

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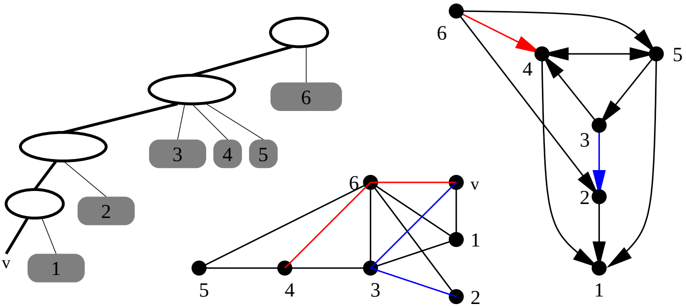

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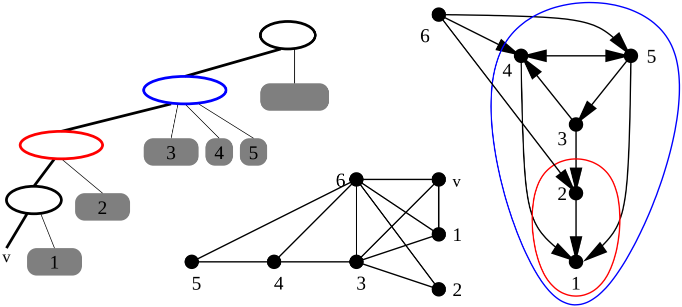

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Complexity

- [Ehrenfeucht et al.'94] gives a $O(n^2)$ complexity.
- [MS00]: simple $O(n + m \log n)$ vertex partitioning algorithm
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Other algorithms

- [CH94] and [MS94]: the first linear algorithms.
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[Spinrad'03] The new [linear time] algorithm [MS99] is currently too complex to describe easily [...] I hope and believe that in a number of years the linear algorithm can be simplified as well.

- [Tedder, Corneil, Habib, P.'08] simple linear time algorithm

Modular decomposition of undirected graphs

Ehrenfeucht et al's modular decomposition algorithm

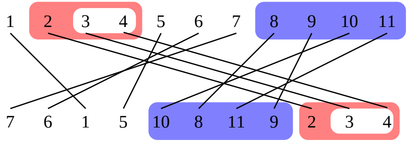
Common Intervals of permutations

Modular decomposition of tournaments

Kernelization algorithm for FAST

Back to permutations: common intervals

In a permutation σ, a set S of consecutive elements is called an interval. Likewise a set S is a common interval of several permutations $\sigma_1, \sigma_2 \ldots$ if it is an interval for every σ_i.



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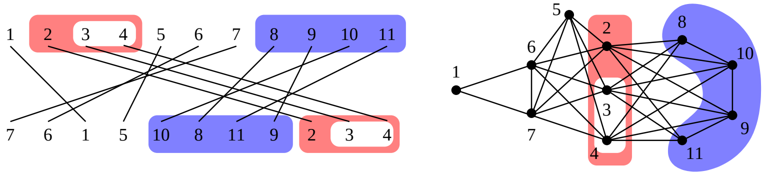

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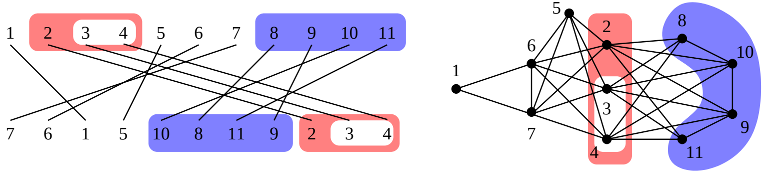

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- Computation of all common intervals in linear time - $O(n^2)$ - [Uno, Yagura'00]

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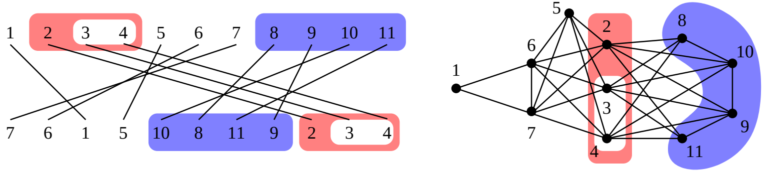

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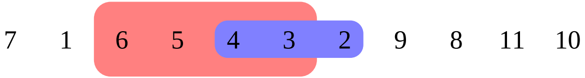
Strong (common) interval doesn't overlap other common intervals

Lemma [de Montgolfier] A set S is a strong interval of σ_1 and σ_2 iff it is a strong module of the permutation graph $G(\sigma_1, \sigma_2)$

Common intervals (2)

The family of common intervals is weakly partitive:



if I_1 and I_2 are two common intervals then

- $I_1 \cup I_2$ is a common interval
- $I_1 \cap I_2$ is a common interval
- $I_1 \setminus I_2$ and $I_2 \setminus I_1$ are common intervals

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Theorem Let σ be a permutation on $[1, \ldots n]$ and $\mathcal{I}$ be the partition into maximal common intervals of σ, then either

1. $\sigma_{/\mathcal{I}} = \mathbb{1}_{|\mathcal{I}|}$ - the identity on $[1 \ldots |\mathcal{I}|]$
2. $\sigma_{/\mathcal{I}} = \overline{\mathbb{1}_{|\mathcal{I}|}}$ - the reverse identity on $[1 \ldots |\mathcal{I}|]$
3. $\sigma_{/\mathcal{I}}$ is prime - or simple (does not have non-trivial common interval)

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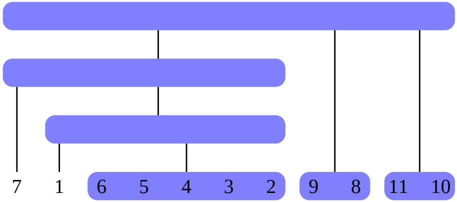
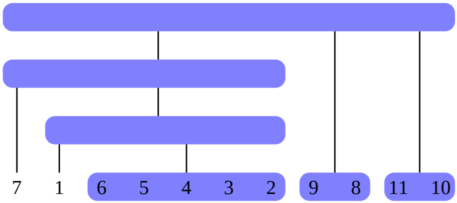


Theorem (see e.g. [Bergeron et al.'08])
The common interval tree can be computed in $O(n)$ time.

Common intervals (4)

A permutation σ is separable if it does not contains the pattern

3 1 4 2



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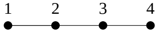
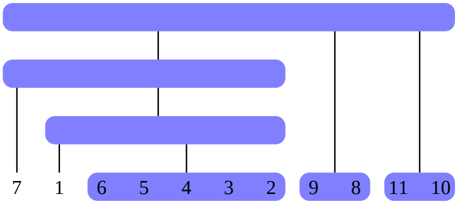
- 3 1 4 2 corresponds to the P_4 



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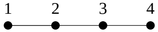
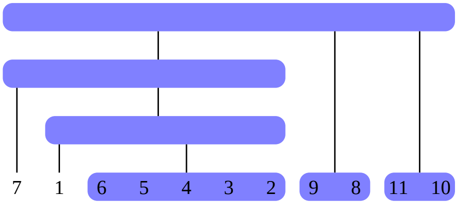


- a permutation is separable iff
  its common interval tree does not contains prime nodes
- a permutation is separable iff
  the permutation graph $G(\sigma, \mathbb{1})$ is a cograph (P_4-free graph)



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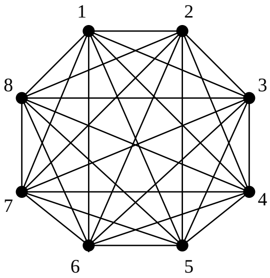

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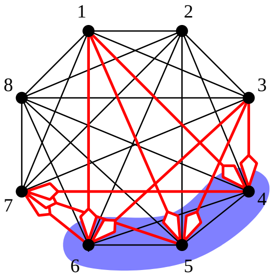
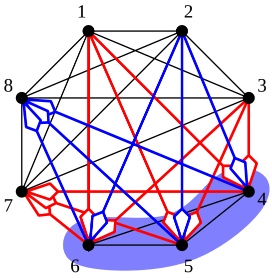


A module in a tournament is a set S such that for every $x \notin S$

- either $\forall y \in S,\ x \rightarrow y$ or $\forall y \in S,\ y \rightarrow x$

Modular decomposition of tournaments



A tournament is transitive if there exists a permutation σ of $V(T)$ with no backward arcs

Theorem: Let T be a tournament and $\mathcal{M}(T)$ be the modular partition into maximal strong modules, then

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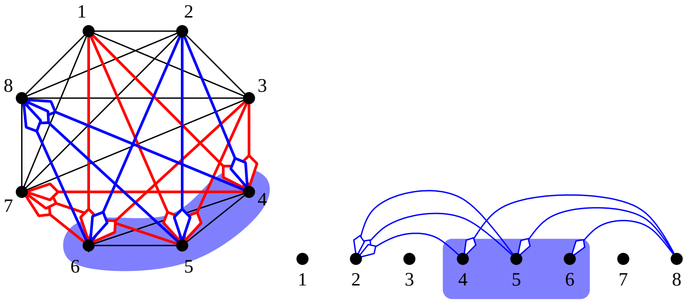

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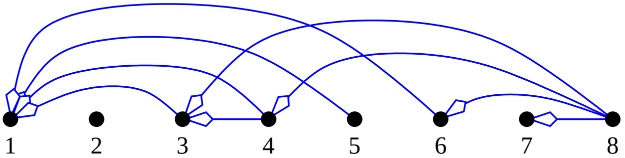

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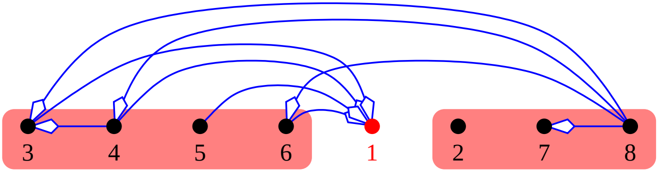

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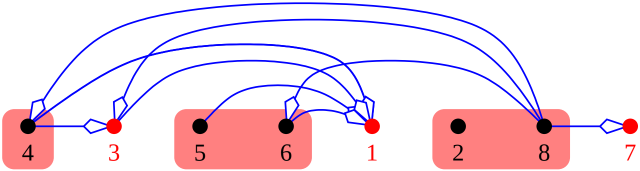

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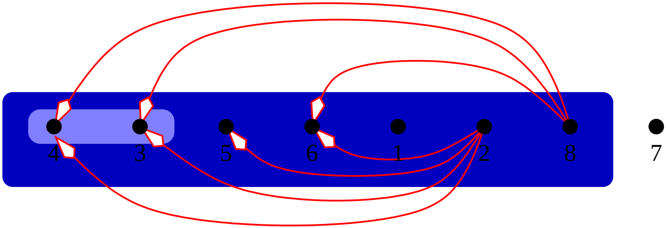

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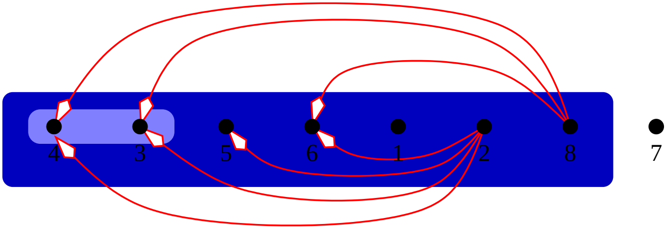

- Modular decomposition tree from a factoring permutation in linear time [Capelle'97]

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FAST: Feedback Arc Set in Tournament

- A tournament T and an integer k
- Find a set of at most k arcs whose reversal transform T into a transitive tournament

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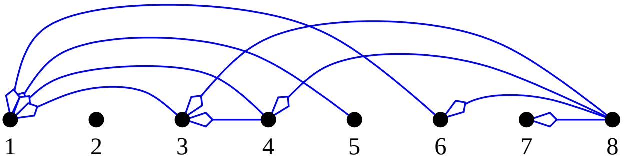

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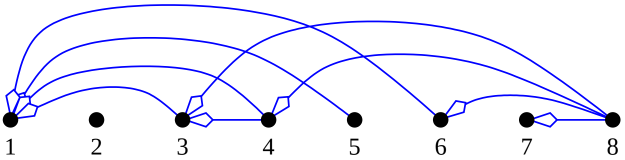
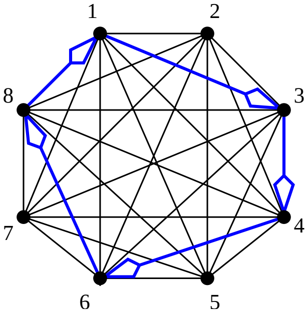


- NP-Complete [Alon'06] [Charbit et al.'07]
- FTP [Raman, Saurabh'06] [Alon et al.'09]
- $(1+\epsilon)$-approximation scheme [Kenyon-Mathieu, Schudy'07]

FAST (2)

Obs.: A tournament is transitive iff there is no (directed) triangle



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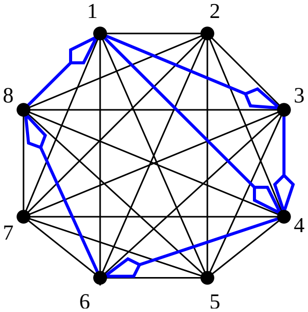

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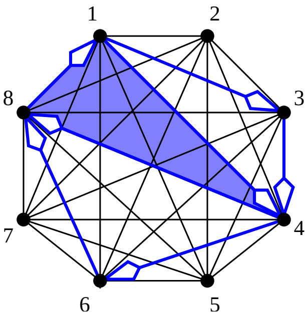

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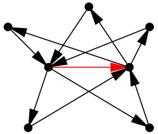
Rule 1 [irrelevant vertex] If a vertex v is not contained in any triangle, then delete v

Rule 2 [sunflower] If there is an arc belonging to more that k distinct triangles, then reverse it and decrease k by 1



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A reduced tournament contains no source nor sink

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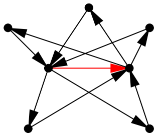
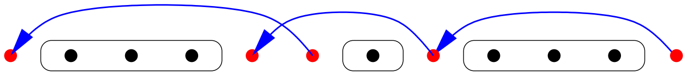


The span $s(\overrightarrow{uv})$ of a backward arc of a reduced tournament is $\leqslant 2k+2$



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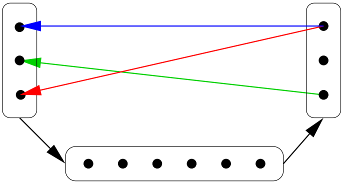
Rule 3 [acyclic module] Let M be a maximal acyclic module. If there are at most $p = |M|$ arcs from $N^+(M)$ to $N^-(M)$, then reverse all these arcs and decrease k by p.



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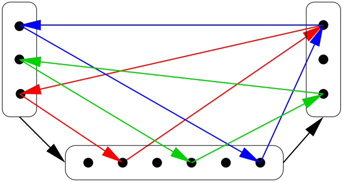

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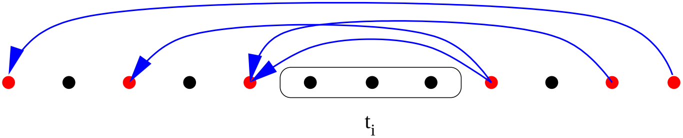

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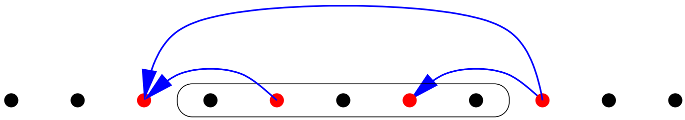

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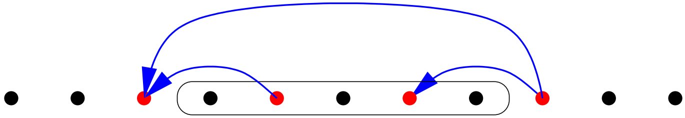
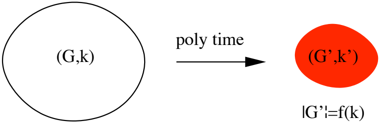


Theorem [Bessy et al.'09]: Every instance (T, k) of k-FAST can be reduced in polynomial time to an equivalent instance (T, k') such that

$$|T| \leqslant 2k + \sum t_i = O(k\sqrt{k}) \text{ and } k' \leqslant k$$

Kernelization algorithm



Given a parameterized instance $(\mathcal{I}, k)$ of a problem, a kernelization algorithm computes in polytime an equivalent instance $(\mathcal{I}', k')$ st.

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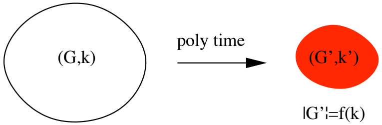

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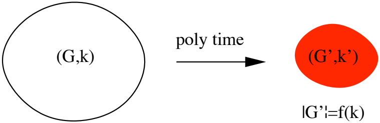
- we described a $O(k\sqrt{k})$-vertex kernel for FAST based on modular decomposition
- best known result: $O(k)$-vertex kernel ([Bessy et al.'09], [P., Perez, Thomassé'11])

Kernelization algorithm



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Other modular decomposition kernelizations:

- $O(k^2)$-vertex kernel for CLUSTER EDITING
- $O(k^3)$-vertex kernel for COGRAPH EDITING
- also for MIN FLIP CONSENSUS TREE, CLOSEST 3-LEAF POWER...

Some conclusions

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  - modules in undirected graphs, digraphs, hypergraphs
  - common interval of permutations
- Various generalizations
  - bimodular decomposition (module adapted to bipartite graphs)
  - bipartitive families : eg. split decomposition of graphs - $O(n + \alpha(n, m).m)$ circle graph recognition
  - crossing families, union-difference families of sets...
  - clique-width (cographs are clique-widht 2 graphs), rankwidth

To learn / read more

- Habib and P. "A survey of the algorithmic aspects of modular decomposition" in Computer Science Review 4:41-59, 2010

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- F. de Montgolfier, "Décomposition modulaire de graphes, théorie, extensions et algorithmes", Phd Thesis (in French), 2003
- B.M. Bui Xuan, "Tree-representation of set families in graph decompositions and efficient algorithms", Phd Thesis, 2008

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- F. de Montgolfier, "Décomposition modulaire de graphes, théorie, extensions et algorithmes", Phd Thesis (in French), 2003
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- C. Crespelle, "Représentations dynamiques de graphes", PhD Thesis (in French), 2007 attend I. Todinca's talk !!!

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